PRE-CALCULUS

By: Salah Abed, Sonia Farag,
Stephen Lane, Tyler Wallace, and Barbara Whitney

MATH 141/142

Selected exercises, examples, and lessons were remixed from *PreCalculus: An Investigation of Functions* by David Lippman and Melonie Rasmussen under a CC-BY-SA license.

Acknowledgments

We would like to thank Big Bend Community College President, Dr. Terry Leas, the Board of Trustees, Bill Autry and the rest of the Exceptional Faculty Award Committee at Big Bend Community College for their generous support that made this text possible.

Edits By

April Mayer
Sarah Adams

Table of Contents

MATH 141 Pre-Calculus I

MATH 142 Pre-Calculus II

Chapter 1:

College Algebra

1.1 Simplifying with Exponents

Exponential Notation:

Let a be a real number, variable, or algebraic expression, and let n be a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

where n is the *exponent* and a is the *base*. The expression a^n is read "a to the n-th power".

Properties of Exponents:

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be rational numbers (assume all denominators and bases are nonzero).

Example	Property
$7^0 = 1, \left(-\sqrt{3}\right)^0 = 1$	$a^0 = 1$
$3^{-5} = \frac{1}{3^5}, (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$	$a^{-n} = \frac{1}{a^n}$
$3^2 \cdot 3^3 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243$	$a^m \cdot a^n = a^{m+n}$
$(2^2)^4 = (2 \cdot 2)^4 = 2^4 \cdot 2^4 = 2^8 = 256$	$(a^m)^n = a^{mn}$
$(10)^3 = (2 \cdot 5)^3 = 2^3 \cdot 5^3 = 8 \cdot 125 = 1000$	$(ab)^n = a^n b^n$
$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
$\frac{2^7}{2^3} = 2^{7-3} = 2^4 = 16$ $\frac{2^3}{2^7} = \frac{1}{2^{7-3}} = \frac{1}{2^4} = 1/16$	if $m > n$ then $\frac{a^m}{a^n} = a^{m-n}$ if $m < n$ then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$
$\frac{2^{-3}}{3^{-4}} = \frac{\frac{1}{2^3}}{\frac{1}{3^4}} = \frac{1}{2^3} \cdot \frac{3^4}{1} = \frac{3^4}{2^3} = \frac{81}{8}$	$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$
$\left(\frac{2}{3}\right)^{-3} = \frac{2^{-3}}{3^{-3}} = \frac{3^3}{2^3} = \frac{27}{8}$	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Simplifying Expressions Containing Exponents:

Use the properties of exponents to simplify each expression:

Example 1: Simplify

$(4x^3y^5)(3x^2y^4)$	Rearrange factors
$(3)(4)x^3x^2y^5y^4$	Multiply numbers, add exponents
$12x^5y^9$	Final answer

Example 2: Simplify

$(4a^2b^3c)^{1/2}$	½ as exponent on each factor
$4^{1/2}a^{2/2}b^{3/2}c^{1/2}$	Simplify, recall $4^{1/2} = \sqrt{4} = 2$
$2ab^{3/2}c^{1/2}$	Final answer

Example 3: Simplify

$\left(\frac{2r^2}{5}\right)^3\left(\frac{5}{r^3}\right)^3$	Exponents on numerator and denominator
$\frac{(2r^2)^3}{5^3}\cdot\frac{5^3}{(r^3)^3}$	Exponents on each factor in parentheses
$\frac{2^3r^6}{5^3}\cdot\frac{5^3}{r^9}$	Divide out the 5^3, simplify $2^3 = 8$
$\frac{8r^6}{r^9}$	Subtract exponents, denominator is larger
$\frac{8}{r^3}$	Final answer

Example 4: Simplify

$(u^{-3}v^2)^{-3}$ Exponent on each factor in parentheses

$(u^{-3})^{-3}(v^2)^{-3}$ Multiply exponents

u^9v^{-6} Move negative exponent

$\dfrac{u^9}{v^6}$ Final answer

Example 5: Simplify

$\dfrac{9x^2y^{-4}}{3x^{-1}y^2}$ Rearrange with negative exponents in one fraction

$\dfrac{9x^2}{3y^2}\cdot\dfrac{y^{-4}}{x^{-1}}$ Move negative exponents and make positive

$\dfrac{9x^2}{3y^2}\cdot\dfrac{x^1}{y^4}$ Add exponent, reduce numbers

$\dfrac{3x^3}{y^6}$ Final answer

Example 6: Simplify

$\left(\dfrac{u^3}{8v}\right)^{-1/3}$ Factor $8 = 2^3$, negative exponent switches the fraction

$\left(\dfrac{2^3v}{u^3}\right)^{1/3}$ Exponent on each factor in parentheses

$\dfrac{(2^3)^{1/3}v^{1/3}}{(u^3)^{1/3}}$ Multiply exponents

$\dfrac{2v^{1/3}}{u}$ Final answer

Example 7: Simplify

$(2x^{1/3} - y^{1/3})(4x^{2/3} + 2x^{1/3}y^{1/3} + y^{2/3})$ Multiply $2x^{1/3}$ and $-y^{1/3}$ by trinomial

$8x + 4x^{2/3}y^{1/3} + 2x^{1/3}y^{2/3} - 4x^{2/3}y^{1/3} - 2x^{1/3}y^{2/3} - y$ Combine like terms

$8x - y$ Final answer

Example 8: Simplify

$x^n x^{3n-1}(x^{2n+3})^2$ Multiply exponent through parenthesis

$x^n x^{3n-1} x^{4n+6}$ Add exponents

x^{8n+5} Final answer

Example 9: Simplify

$\frac{(3x^{k+3})^2}{x^{2(k+1)}} \cdot \frac{x^k}{(x^k)^3}$ Multiply exponents through parenthesis

$\frac{3^2 x^{2k+6}}{x^{2k+2}} \cdot \frac{x^k}{x^{3k}}$ Add exponents in numerator and denominator

$\frac{9x^{3k+6}}{x^{5k+2}}$ Subtract exponents, denominator is larger

$\frac{9}{x^{2k-4}}$ Final answer

Example 10: Simplify

$$\left[\left(\frac{x^a}{y^b}\right)^3\left(\frac{x^{2a}}{y^{3b}}\right)^{-3}\right]^{-4}$$ 3 and -3 as exponents on each factor

$$\left[\frac{x^{3a}}{y^{3b}}\cdot\frac{x^{-6a}}{y^{-9b}}\right]^{-4}$$ Add exponents

$$\left(\frac{x^{-3a}}{y^{-6b}}\right)^{-4}$$ -4 as exponent on each factor

$$\frac{x^{12a}}{y^{24b}}$$ Final answer

Example 11:

$$\left(\frac{x^{4k+1}y^{-k-1}}{x^{-2k+1}y^{-10k-1}}\right)^{\frac{1}{3k}}$$ Subtract exponents

$$(x^{6k}y^{9k})^{1/3k}$$ $\frac{1}{3k}$ as exponent on each factor

$$x^2y^3$$ Final answer

Example 12:

$$\left(\frac{x^{1/2n}x^{1/6n}}{x^{-1/3n}}\right)^{\frac{n}{2}}$$ Add exponents in numerator

$$\left(\frac{x^{2/3n}}{x^{-1/3n}}\right)^{\frac{n}{2}}$$ Subtract exponents in denominator

$$\left(x^{1/n}\right)^{n/2}$$ $\frac{n}{2}$ as exponent on the factor

$$x^{1/2}$$ Final answer

Example 13:

$$\left(\frac{x^{2k+1}y^{k+1}}{x^{3k+1}y^{1-k}}\right) \div \left(\frac{x^{3(k+1)}y^{3-3k}}{x^{3(k-1)}y^{-3k}}\right)^{\frac{k}{2}}$$ Distribute through parenthesis in exponents

$$\left(\frac{x^{2k+1}y^{k+1}}{x^{3k+1}y^{1-k}}\right) \div \left(\frac{x^{3k+3}y^{3-3k}}{x^{3k-3}y^{-3k}}\right)^{\frac{k}{2}}$$ Subtract exponents

$$\left(\frac{y^{2k}}{x^{k}}\right) \div (x^6y^3)^{k/2}$$ $\frac{k}{2}$ as exponent on each factor

$$\left(\frac{y^{2k}}{x^{k}}\right) \div (x^{3k}y^{3k/2})$$ Multiply by reciprocal

$$\left(\frac{y^{2k}}{x^{k}}\right) \cdot \left(\frac{1}{x^{3k}y^{3k/2}}\right)$$ Add exponents for multiplying, subtract exponents for dividing

$$\frac{y^{k/2}}{x^{4k}}$$ Final answer

1.1 Simplifying with Exponents Practice

Simplify the following expressions

1. $(-3x^3y^{-2}z)^{-2}$

2. $\left(\sqrt[6]{5}x^{7/4}y^{-2/3}\right)^{12}$

3. $\left(\frac{x^{-2}y^6}{9}\right)^{-1/2}$

4. $\frac{a^{-2/3}b^{1/2}}{b^{-2/3}\sqrt[3]{a}}$

5. $3\left(\frac{9}{4}\right)^{3/2} \cdot 4\sqrt[3]{81} \cdot 243^{-1/5}$

6. $(4m^2n^{1/2}r)(-2mn^{2/3}r^{-1})$

7. $(8x^3y^{3/2})^{2/3}$

8. $\left(\frac{x^{-1}y}{x^{1/2}y^{-2/3}}\right) \div \left(\frac{x^{-3}}{y^{-1}}\right)^{-1/2}$

9. $\left(\frac{z^{-2/3}}{5^{-1}z^{1/3}}\right)^{-2}$

10. $(3a^2b^{2/3}c^3)(-2a^5b^{1/3}c^{-2})$

11. $\left(\frac{a^{1/2}b^{2/3}}{2^{-1}c^{-2}}\right)^6$

12. $(-4x^2y^{5/2})\left(\frac{x^{-1}y^{-1/2}}{-2}\right)$

13. $(125x^2y^{1/3}a)^{2/3}(-2xy^{1/6}a^{-2/3})$

14. $\left(\frac{3x^{2/3}y^3}{2x^{5/2}z^{-3}}\right)\left(\frac{16x^4y^5}{x^2z^4}\right)^{1/2}$

15. $\left(\frac{m^2p}{64m^{-3}p^{1/3}}\right)^{-1/3}$

16. $\left(\frac{8y^{1/3}y^{-1/4}}{y^{-1/12}}\right)^2$

17. $\left(\frac{9x^{1/3}x^{1/2}}{x^{-1/6}}\right)^{1/2}$

18. $\left(\frac{16x^{1/3}y^{-2/3}}{9x^{-2/3}y^{-1/3}}\right)^{3/2}\left(\frac{8x^{-3/2}y^{3/2}}{27x^{1/2}y^{-5/2}}\right)^{-2/3}$

19. $\left(\frac{x^{3/4}y^{-3/2}}{x^{-1/4}y^{-3/2}}\right)^{1/2} \div \left(\frac{x^{-1/2}y^{-1/2}}{x^{1/2}y^{-3/2}}\right)^{3/2}$

20. $x^kx^{2k+1}(x^{3k+2})^2$

21. $\frac{(x^{3k+2}x^{4k-3})^2}{x^{14k}}$

22. $\frac{(a^kb^{k+1})^2}{(a^{2+k}b^{3+k})^2}$

23. $\frac{(x^{2k+3}y^{3k+11})^3}{(x^{k+1}y^{k+4})^6}$

24. $\frac{x^{2n-3}}{x^{3n+1}} \cdot \frac{x^{n+5}}{x^{n-2}}$

25. $\dfrac{(2x^{n+1})^2}{x^{2(n+1)}} \cdot \dfrac{x^{3-n}}{(x^n)^2}$

26. $(c^n - k^{3n})(c^{2n} + c^n k^{3n} + k^{6n})$

27. $\dfrac{(3x^{n+1})^2}{x^{2(n+1)}} \cdot \dfrac{x^n}{(x^n)^3}$

28. $\dfrac{a^{6n+1}b^{5n-2}}{a^{6n-1}b^{5n+2}}$

29. $\dfrac{(3a^k b^{4k})^3}{(5a^{k+1}b^{2k+1})^2}$

30. $a^{2n} \cdot a^{n+1}$

31. $(a^n - 1)(a^{2n} + 1)$

32. $(x^n + 1)^2$

33. $\dfrac{x^{2n+1}}{x^n y}$

34. $(x^a y^b \cdot x^b y^a)^c$

35. $(m^{x-b} \cdot n^{x+b})^x (m^b n^{-b})^x$

36. $\left[\dfrac{(3x^a y^b)^3}{(-3x^a y^b)^2}\right]^2$

37. $\left[\left(\dfrac{x^r}{y^t}\right)^2 \left(\dfrac{x^{2r}}{y^{4t}}\right)^{-2}\right]^{-3}$

38. $\left(\dfrac{x^{5n-1}y^{-n+1}}{x^{-3n-1}y^{-5n+1}}\right)^{\frac{1}{2n}}$

39. $\left(\dfrac{a^{-4(n+2)}b^{3n-6}}{a^{(2n-8)}b^{-3(n+2)}}\right)^{\frac{1}{3n}}$

40. $\left(\dfrac{y^{1/3n}y^{-1/4n}}{y^{-1/12n}}\right)^{2n}$

41. $\left(\dfrac{x^{1/3n}x^{1/2n}}{x^{-1/6n}}\right)^{\frac{n}{2}}$

42. $\left(\dfrac{x^{2n-1}y^{3-n}}{x^{3n-1}y^{3-2n}}\right)^{\frac{1}{n}}$

43. $\left(\dfrac{x^{2(n-1)}y^{3-n}}{x^{3n-2}y^{3-2n}}\right)^{\frac{2}{n}}$

44. $\left(\dfrac{x^{2(2n+1)}y^{2-n}}{x^{2(n+1)}y^{2+n}}\right)^{-\frac{1}{2}}$

45. $\dfrac{x^{2n+1}y^{n+1}}{x^{3n+1}y^{1-n}} \div \left(\dfrac{x^{n+1}y^{1-n}}{x^{n-1}y^{-n}}\right)^n$

46. $\left(\dfrac{x^{2n+1}y^{n+1}}{x^{3n+1}y^{1-n}}\right) \div \left(\dfrac{x^{2(n+1)}y^{2(1-n)}}{x^{2(n-1)}y^{-2n}}\right)^{\frac{n}{2}}$

47. $(x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3})$

48. $(x^{1/3} + y^{1/3})(x^{2/3} - x^{1/3}y^{1/3} + y^{2/3})$

49. $(x^{1/2} - 4x^{1/4} - 4)(x^{1/2} + 4x^{1/4} - 4)$

50. $(x^{1/2} - 2x^{1/4}y^{1/4} - y^{1/2})(x^{1/2} + 2x^{1/4}y^{1/4} - y^{1/2})$

51. $(x^{1/2} - 2x^{1/4}y^{1/4} + y^{1/2})(x^{1/2} + 2x^{1/4}y^{1/4} + y^{1/2})$

52. $(x^{1/2} - 2x^{1/4}y^{1/4} - 2y^{1/2})(x^{1/2} + 2x^{1/4}y^{1/4} - 2y^{1/2})$

53. $(x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^3y^{n-4} + x^2y^{n-3} + xy^{n-2} + y^{n-1})$

1.2 Radical Expressions and Equations

Principle $\boldsymbol{n}$-th root $\sqrt[n]{\boldsymbol{a}}$:

Let n be a positive integer greater than 1, and let a be a real number.

(1) If $a = 0$ then $\sqrt[n]{a} = 0$

(2) If $a > 0$ then $\sqrt[n]{a}$ is the positive real number b such that $b^n = a$

(3) (a) If $a < 0$ and n is odd, then $\sqrt[n]{a}$ is the negative real number b such that $b^n = a$

(b) If $a < 0$ and n is even, then $\sqrt[n]{a}$ is not a real number.

The expression $\sqrt[n]{a}$ is a *radical*, the number a is the *radicand*, and n is the *index* of the radical. The symbol $\sqrt{\ }$ is called a *radical sign*.

If $\sqrt{a} = b$, then $b^2 = a$. If $\sqrt[3]{a} = b$, then $b^3 = a$

Properties of $\sqrt[n]{\boldsymbol{a}}$: (where n is a positive integer)

Example	Property
$\left(\sqrt{7}\right)^2 = 7, \left(\sqrt[3]{-5}\right)^3 = -5$	$\left(\sqrt[n]{a}\right)^n = a$ if $\sqrt[n]{a}$ is a real number
$\sqrt{3^2} = 3, \sqrt[3]{2^3} = 2$	$\sqrt[n]{a^n} = a$ if $a \geq 0$
$\sqrt[3]{(-2)^3} = -2, \sqrt[5]{(-3)^5} = -3$	$\sqrt[n]{a^n} = a$ if $a < 0$ and n is odd
$\sqrt{(-2)^2} = \lvert -2 \rvert = 2, \sqrt[4]{(-3)^4} = \lvert -3 \rvert = 3$	$\sqrt[n]{a^n} = \lvert a \rvert$ if $a < 0$ and n is even

$$\sqrt{x^2} = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Laws of Radicals:

Example	Law
$\sqrt{98} = \sqrt{49 \cdot 2} = \sqrt{49} \cdot \sqrt{2} = 7\sqrt{2}$ $\sqrt[3]{-54} = \sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
$\sqrt[3]{\frac{3}{8}} = \frac{\sqrt[3]{3}}{\sqrt[3]{8}} = \frac{\sqrt[3]{3}}{2}$	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
$\sqrt{\sqrt[3]{64}} = \sqrt[(2)(3)]{64} = \sqrt[6]{64} = \sqrt[6]{2^6} = 2$	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Warning: If $a \neq 0$ and $b \neq 0$

Warning	Example
$\sqrt{a^2 + b^2} \neq a + b$	$\sqrt{3^2 + 4^2} = \sqrt{25} = 5 \neq 3 + 4 = 7$
$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$	$\sqrt{4 + 9} = \sqrt{13} \neq \sqrt{4} + \sqrt{9} = 2 + 3 = 5$

Simplifying Radicals

An expression involving radicals is in simplest form when the following conditions are satisfied:

1. All possible factors have been removed from the radical
2. All fractions have radical-free denominators (accomplished by a process called rationalizing the denominator)
3. The index of the radical is reduced.

To simplify a radical, we factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical and the “leftover” factors make up the new radicand.

Example 1: Simplify

$\sqrt[3]{135}$	Factor 135
$\sqrt[3]{27 \cdot 5}$	Split into two radicals
$\sqrt[3]{3^3}\sqrt[3]{5}$	Simplify the first radical
$3\sqrt[3]{5}$	Final answer

Example 2: Simplify

$\sqrt[3]{16a^3b^8c^4}$	Factor into multiples of the index
$\sqrt[3]{(2^3a^3b^6c^3)(2b^2c)}$	Divide a 3 (the index) out of each exponent
$\sqrt[3]{(2ab^2c)^3(2b^2c)}$	Split into two radicals
$\sqrt[3]{(2ab^2c)^3}\sqrt[3]{2b^2c}$	Simplify first radical
$2ab^2c\sqrt[3]{2b^2c}$	Final answer

Example 3: Simplify

$\sqrt{3x^2y^3}\sqrt{6x^5y}$	Multiply radicals together, factor the 6
$\sqrt{3x^2y^3 \cdot 2 \cdot 3x^5y}$	Combine bases
$\sqrt{3^2 \cdot 2x^7y^4}$	Factor into multiples of the index
$\sqrt{(3^2x^6y^4)(2x)}$	Divide a 2 (the index) out of each exponent
$\sqrt{(3x^3y^2)^2(2x)}$	Split into two radicals
$\sqrt{(3x^3y^2)^2}\sqrt{2x}$	Simplify the first radical
$3x^3y^2\sqrt{2x}$	Final answer

Rationalizing Denominators of Quotients ($a > 0$)

Factor in denominator	Multiply numerator and denominator by	Resulting factor
$\sqrt{x}$	$\sqrt{x}$	$\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$
$\sqrt[3]{x}$	$\sqrt[3]{x^2}$	$\sqrt[3]{x}\sqrt[3]{x^2} = x$
$\sqrt[7]{x^3}$	$\sqrt[7]{x^4}$	$\sqrt[7]{x^3}\sqrt[7]{x^4} = x$

Example 4: Simplify

$$\frac{1}{\sqrt{3}}$$ Multiply numerator and denominator by $\sqrt{3}$

$$\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$ Multiply

$$\frac{\sqrt{3}}{\sqrt{3^2}}$$ Simplify denominator

$$\frac{\sqrt{3}}{3}$$ Final answer

Example 5: Simplify

$$\frac{1}{\sqrt[3]{a}}$$ Multiply numerator and denominator by $\sqrt[3]{a^2}$

$$\frac{1}{\sqrt[3]{a}}\left(\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}\right)$$ Multiply

$$\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^3}}$$ Simplify denominator

$$\frac{\sqrt[3]{a^2}}{a}$$ Final answer

Example 6: Simplify

$$\sqrt{\frac{3}{2}}$$ Square root of numerator and denominator

$$\frac{\sqrt{3}}{\sqrt{2}}$$ Multiply numerator and denominator by $\sqrt{2}$

$$\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$ Multiply

$$\frac{\sqrt{6}}{\sqrt{2^2}}$$ Simplify denominator

$$\frac{\sqrt{6}}{2}$$ Final answer

Example 7: Simplify

$$\sqrt[5]{\frac{x}{y^2}}$$ Take root of numerator and denominator

$$\frac{\sqrt[5]{x}}{\sqrt[5]{y^2}}$$ Multiply numerator and denominator by $\sqrt[5]{y^3}$

$$\frac{\sqrt[5]{x}}{\sqrt[5]{y^2}}\left(\frac{\sqrt[5]{y^3}}{\sqrt[5]{y^3}}\right)$$ Multiply

$$\frac{\sqrt[5]{xy^3}}{\sqrt[5]{y^5}}$$ Simplify denominator

$$\frac{\sqrt[5]{xy^3}}{y}$$ Final answer

Definition of Rational Exponents:

Let $\frac{m}{n}$ be a rational number where n is a positive integer greater than 1. If a is a real number such that $\sqrt[n]{a}$ exists, then

(1) $a^{1/n} = \sqrt[n]{a}$

(2) $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

(3) $a^{m/n} = \left(a^{1/n}\right)^m = (a^m)^{1/n}$

Example 8: Simplify

$(-125)^{2/3}(4)^{-5/2}$	Convert to radicals
$\left(\sqrt[3]{(-5)^3}\right)^2 \left(\sqrt{4}\right)^{-5}$	Take roots
$(-5)^2(2)^{-5}$	Move negative exponent and make it positive
$\frac{(-5)^2}{2^5}$	Evaluate exponents
$\frac{25}{32}$	Final answer

Example 9: Simplify

$(a^2b^9)^{1/3}$	Put exponent on each factor
$(a^2)^{1/3}(b^9)^{1/3}$	Multiply exponents
$a^{2/3}\, b^3$	Final answer

Example 10: Simplify

$\left(\frac{2x^{2/3}}{y^{1/2}}\right)^2\left(\frac{3x^{-1/6}}{y^{1/3}}\right)$ Put exponent on each factor, squaring 2 to get 4

$\left(\frac{4x^{4/3}}{y}\right)\left(\frac{3x^{-1/6}}{y^{1/3}}\right)$ Multiply bases, add exponents

$\frac{(4\cdot 3)x^{4/3-1/6}}{y^{1+1/3}}$ Common denominators on exponents

$\frac{12x^{8/6-1/6}}{y^{3/3+1/3}}$ Add exponents

$\frac{12x^{7/6}}{y^{4/3}}$ Final answer

Addition and Subtraction with Radicals

We add or subtract radicals just as we add or subtract like terms

Example 11: Add

$7\sqrt{5}+3\sqrt{5}$ Add like radicals, radical remains unchanged

$10\sqrt{5}$ Final answer

Example 12: Add

$9\sqrt[3]{2}-7x\sqrt[3]{2}+4\sqrt[3]{2}$ Add like radicals, note $7x\sqrt[3]{2}$ is not like others

$13\sqrt[3]{2}-7x\sqrt[3]{2}$ Final answer

Example 13: Add

$8\sqrt[5]{4x}+2\sqrt[5]{4x}-\sqrt[3]{4x}$ Add like radicals, note index must match

$10\sqrt[5]{4x}-\sqrt[3]{4x}$ Final answer

Example 14: Add

$$5\sqrt[3]{16y^4} + 7y\sqrt[3]{2y}$$ Factor first radical

$$5\sqrt[3]{2^3y^3}\sqrt[3]{2y} + 7y\sqrt[3]{2y}$$ Simplify first radical

$$5 \cdot 2y\sqrt[3]{2y} + 7y\sqrt[3]{2y}$$ Multiply

$$10y\sqrt[3]{2y} + 7y\sqrt[3]{2y}$$ Add like terms

$$17y\sqrt[3]{2y}$$ Final answer

Radical Equations

The principle of powers: If an equation $a = b$ is true, then $a^n = b^n$ is true for any rational number n for which a^n and b^n exist. This means we will have to check our solutions in the original equation to be sure they work. If a value does not work it is called an extraneous solution and not included in the final answer

Example 15: Solve

$$\sqrt{5x+3} = 7$$ Square both sides

$$5x + 3 = 49$$ Subtract 3 from both sides

$$5x = 46$$ Divide both sides by 5

$$x = \frac{46}{5}$$ Check solution

$$\sqrt{5\left(\frac{46}{5}\right) + 3} = 7$$
$$\sqrt{46+3} = 7$$
$$\sqrt{49} = 7$$
$$7 = 7$$ It works!

$$x = \frac{46}{5}$$ Final answer

Example 16: Solve

$\sqrt{4x-9}-\sqrt{2x}=0$ Add $\sqrt{2x}$ to isolate one radical term

$\sqrt{4x-9}=\sqrt{2x}$ Square both sides

$4x-9=2x$ Subtract $2x$ and add 9 to both sides

$2x=9$ Divide by 2

$x=\frac{9}{2}$ Check solution

$$\sqrt{4\left(\frac{9}{2}\right)-9}-\sqrt{2\left(\frac{9}{2}\right)}=0$$
$$\sqrt{18-9}-\sqrt{9}=0$$
$$\sqrt{9}-3=0$$
$$3-3=0$$
$$0=0$$

It works!

$x=\frac{9}{2}$ Final answer

Example 17: Solve

$\sqrt{3-x}-x=3$	Add x to isolate the radical
$\sqrt{3-x}=x+3$	Square both sides
$3-x=x^2+6x+9$	Subtract 3 and add x
$0=x^2+7x+6$	Factor
$0=(x+6)(x+1)$	Set each factor equal to zero
$x+6=0$ or $x+1=0$	Solve both equations
$x=-6,-1$	Check both solutions
$\sqrt{3-(-6)}-(-6)=3$ $\sqrt{9}+6=3$ $3+6=3$ $9\neq 3$	Checking $x=-6$ Extraneous solution, not included in final answer
$\sqrt{3-(-1)}-(-1)=3$ $\sqrt{4}+1=3$ $2+1=3$ $3=3$	Checking $x=-1$ It works!
$x=-1$	Final answer

Example 18: Solve

$\sqrt{4y+1}-\sqrt{y-2}=3$	Adding $\sqrt{y-2}$ isolates one of the radical terms
$\sqrt{4y+1}=3+\sqrt{y-2}$	Square both sides
$\left(\sqrt{4y+1}\right)^2=\left(3+\sqrt{y-2}\right)^2$	Simplify, recall $(a+b)^2=a^2+2ab+b^2$
$4y+1=9+6\sqrt{y-2}+y-2$	Combine like terms
$4y+1=7+y+6\sqrt{y-2}$	Subtract 7 and y to isolate the term with radical
$3y-6=6\sqrt{y-2}$	Divide by common factor of 3
$y-2=2\sqrt{y-2}$	Square both sides
$(y-2)^2=\left(2\sqrt{y-2}\right)^2$	Simplify, recall $(a+b)^2=a^2+2ab+b^2$
$y^2-4y+4=4(y-2)$	Distribute
$y^2-4y+4=4y-8$	Subtract 4y, add 8 to make equal to zero
$y^2-8y+12=0$	Factor
$(y-6)(y-2)=0$	Set each factor equal to zero
$y-6=0$ or $y-2=0$	Solve
$y=6,2$	We need to check these answers
$\sqrt{4(6)+1}-\sqrt{(6)-2}=3$ $\sqrt{24+1}-\sqrt{6-2}=3$ $\sqrt{25}-\sqrt{4}=3$ $5-2=3$ $3=3$	Checking $y=6$ It works!
$\sqrt{4(2)+1}-\sqrt{(2)-2}=3$ $\sqrt{8+1}-\sqrt{2-2}=3$ $\sqrt{9}-\sqrt{0}=3$ $3-0=3$ $3=3$	Checking $y=2$ It works!
$y=6,2$	Final answer

1.2 Radical Expressions and Equations Practice

Simplify each expression.

1. $\sqrt{50} + 2\sqrt{8}$

2. $\sqrt{63} - 2\sqrt{7} + \sqrt{27}$

3. $2\sqrt[3]{2} - \sqrt[3]{54} + \sqrt[3]{250}$

4. $x\sqrt{2x} + 4\sqrt{18x^3}$

5. $\sqrt{\frac{1}{2}} + \sqrt{8}$

6. $\sqrt{48} - 2\sqrt{\frac{1}{3}}$

7. $\sqrt{75} + 4\sqrt{18} + 2\sqrt{12} - 2\sqrt{8}$

8. $3\sqrt{2a^3} + a\sqrt{18a} - 2\sqrt{8a^3}$

9. $\sqrt{125} + 2\sqrt{27} - \sqrt{20} + 3\sqrt{12}$

10. $2\sqrt{12x} - 3\sqrt{\frac{1}{3}x}$

11. $\sqrt{3a} + 5\sqrt{27a^3} - a\sqrt{3a}$

12. $\sqrt{\frac{1}{3}} + 3\sqrt{27} - 2\sqrt{12}$

13. $\sqrt{\frac{3}{5}} + 5\sqrt{60} - 3\sqrt{15}$

14. $3\sqrt{50} - 4\sqrt{8} + \sqrt{27} - \sqrt{3}$

15. $\sqrt{x^4y} - x\sqrt{9x^2y} + x^2\sqrt{16y}$

16. $a^2\sqrt{8a^3b} - 2a^3\sqrt{18ab} + 3a\sqrt{50a^5b}$

17. $\sqrt{16ab^3} - \sqrt{9ab^3} + \sqrt{25a^3b^3}$

18. $\frac{\sqrt{25x}}{x} + \sqrt{\frac{9}{x}} - \frac{5}{\sqrt{x}}$

19. $\sqrt{32} + \sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}}$

20. $12\sqrt{\frac{7}{3}} - \sqrt{189}$

21. $\frac{1}{2}\sqrt{\frac{3}{4}} + \frac{1}{2}\sqrt{\frac{1}{3}} - 7\sqrt{75}$

22. $\sqrt{125} + 17\sqrt{\frac{1}{5}} - \left(\frac{5}{4}\right)^{-\frac{1}{2}} + \sqrt[4]{\frac{25}{16}}$

23. $6\sqrt{\frac{3}{2}} - \sqrt{24} + 3\sqrt{\frac{2}{3}}$

24. $2\sqrt{\frac{5}{3}} - \sqrt{60}$

25. $\sqrt[4]{36} + 8(4)^{-3/2} - \sqrt[3]{-27} - \left(\frac{2}{\sqrt{2}}\right)^2$

26. $\sqrt[4]{9} - \sqrt{\frac{1}{3}}$

27. $\frac{x}{\sqrt{x}} - x\sqrt{\frac{1}{x}}$

28. $\sqrt[4]{144} + 3\sqrt[4]{9} - 5\sqrt{48}$

29. $\frac{4}{\sqrt[3]{4}} - \sqrt[6]{4}$

30. $\frac{6}{\sqrt{3}} - 18\sqrt{\frac{1}{3}} + 12^{1/2}$

31. $\sqrt[3]{\frac{5}{4}} + \sqrt[3]{\frac{1}{100}}$

32. $\sqrt[3]{54} + \sqrt{\frac{1}{2}} - \sqrt[3]{250} - \frac{3}{4}\sqrt{\frac{2}{9}}$

33. $\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{6}} - 24^{1/2} - 48^{1/3}$

34. $a^2\sqrt{\frac{b}{a}} - \sqrt{\frac{a^3 b}{4}}$

35. $\sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a-b}{a+b}}$

36. $\sqrt{\frac{a+1}{a-1}} + 2\sqrt{1 - \frac{1}{a^2}} - \sqrt{\left(a - \frac{1}{a}\right)\left(\frac{1}{a}\right)}$

37. $\sqrt{1 - \frac{x}{y}} + 2\sqrt{\frac{y^2 - xy}{y^2}}$

38. $\sqrt{3x^2y^3} - \sqrt{12x^3y} + \sqrt{27x^5y} - \sqrt{75y}$

39. $\frac{6}{\sqrt{3}} - 18\sqrt{\frac{1}{3}} - \frac{1}{6}\sqrt{108} + 12^{1/2} + 3^{3/2} + 5\sqrt{3}$

40. $\sqrt{24} - 6\sqrt{\frac{1}{6}} + \frac{1}{2}\sqrt{96} - \sqrt{66\frac{2}{3}} + \frac{2}{5}\sqrt{\frac{25}{6}}$

41. $\sqrt{\frac{5}{4}} + \frac{1}{\sqrt{3}} + 3(8)^{2/3} - \frac{1}{2}\sqrt{4^3} - \frac{7}{6}\sqrt{27} + \frac{1}{3}(-27)^{2/3} + \sqrt{147}$

Solve each equation.

42. $\sqrt{7x+2} = 4$

43. $\sqrt{2x+3} - 3 = 0$

44. $\sqrt{3x+12} - 6 = 0$

45. $\sqrt{5x+1} - 4 = 0$

46. $\sqrt{3x-8} - \sqrt{x} = 0$

47. $\sqrt{6x-5} - x = 0$

48. $\sqrt{5x+1} - 1 = x$

49. $\sqrt{x+2} - \sqrt{x} = 2$

50. $x + \sqrt{4x+1} = 5$

51. $3 + x = \sqrt{6x+13}$

52. $\sqrt{5x+1} - \sqrt{4x+4} = 0$

53. $x - 1 = \sqrt{7-x}$

54. $\sqrt{6x-8} - \sqrt{3x+4} = 0$

55. $\sqrt{3-3x} - 1 = 2x$

56. $\sqrt{2x+1} - \sqrt{x} = 1$

57. $\sqrt{2x+2} = 3 + \sqrt{2x-1}$

58. $\sqrt{3x-2} - \sqrt{x} = 2$

59. $\sqrt{4x+5} - \sqrt{x+4} = 2$

60. $\sqrt{3x+1} = 4 + \sqrt{x+3}$

61. $\sqrt{3x+4} - \sqrt{x+2} = 2$

62. $\sqrt{x+2} + 5 = \sqrt{3x+3}$

63. $\sqrt{2x+4} - \sqrt{x+3} = 1$

64. $\sqrt{4x+4} - \sqrt{x-2} = \sqrt{2x+3}$

65. $\sqrt{3x+4} - \sqrt{2x+1} = \sqrt{x-3}$

66. $\sqrt{4x+1} + \sqrt{x-1} = \sqrt{7x+2}$

67. $\sqrt{2x-1} + \sqrt{x-1} = x$

68. $\sqrt{6x+2} + \sqrt{2x+6} = \sqrt{15x+17}$

1.3 Quadratic Expressions and Equations

Algebraic Expressions

It is convenient to use letters such as x or y to represent numbers. Such a symbol is called a variable. An algebraic expression is the result of performing a finite number of additions, subtractions, multiplications, divisions, or roots on a collection of variables and real numbers. The following are examples of algebraic expressions:

$$x^3 + 3x^2 - \sqrt{x} - \pi, \qquad \frac{4xy - x}{x + y}, \qquad \sqrt[3]{\frac{7x - 3}{x^5y^{-2} + z}}$$

Polynomials

If x is a variable, then a *monomial* in x is an expression of the form ax^n, where a is a real number and n is a nonnegative integer. A *binomial* is a sum of two monomials, and a *trinomial* is a sum of three monomials.

Definition of a *polynomial*:

A polynomial in x is the sum of the form:

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

Where n is a nonnegative integer and each coefficient a is a real number. If $a_n \neq 0$, then the polynomial is said to have *degree* n. Each expression a_kx^k in the sum is a *term* of the polynomial. The *coefficient* a_k of the highest power of x is called the *leading coefficient* of the polynomial.

Example	Leading Coefficient	Degree
$2x^5 + 3x^4 + (-7)x + 3$	2	5
$x^7 + 8x^2 + (-3)x$	1	7
$-5x^2 + 1$	-5	2
8	8	0

Example 1: Add

$(x^3 - 4x^2 + 5x - 9) + (2x^3 + x^2 - 3x)$	Distribute “+”
$x^3 - 4x^2 + 5x - 9 + 2x^3 + x^2 - 3x$	Combine like terms
$3x^3 - 3x^2 + 2x - 9$	Final answer

Example 2: Subtract

$(x^3 + 5x^2 - 10x + 6) - (2x^3 - 3x - 4)$	Distribute “−“
$x^3 + 5x^2 - 10x + 6 - 2x^3 + 3x + 4$	Combine like terms
$-x^3 + 5x^2 - 7x + 10$	Final answer

Example 3: Multiply

$(7x + 5y)(3x - 2y)$	Use FOIL to multiply
$(7x)(3x) = 21x^2$	F – First terms
$(7x)(-2y) = -14xy$	O – Outside terms
$(5y)(3x) = 15xy$	I – Inside terms
$(5y)(-2y) = -10y^2$	L – Last terms
$21x^2 - 14xy + 15xy - 10y^2$	Combine like terms
$21x^2 + xy - 10y^2$	Final answer

Example 4: Multiply

$(3x - 7)(2x^3 + 3x - 1)$	Distribute $3x$ and -7
$(3x)(2x^3) + (3x)(3x) + (3x)(-1) + (-7)(2x^3) + (-7)(3x) + (-7)(-1)$	Multiply
$6x^4 + 9x^2 - 3x - 14x^3 - 21x + 7$	Combine like terms
$6x^4 - 14x^3 + 9x^2 - 24x + 7$	Final answer

Special Products

Certain products of binomials occur so frequently that you should learn to recognize them

Special Product/Factor Formulas:

Name	Property
Difference of two squares	$(a+b)(a-b) = a^2 - b^2$
Sum of squares	$a^2 + b^2 = Prime$
Perfect square trinomial	$(a+b)^2 = a^2 + 2ab + b^2$ $(a-b)^2 = a^2 - 2ab + b^2$
Sum of cubes	$(a+b)(a^2 - ab + b^2) = a^3 + b^3$
Difference of cubes	$(a-b)(a^2 + ab + b^2) = a^3 - b^3$
Binomial cubes	$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Example 5: Find the product

$(2x+3)^2$	Use perfect square formula
$(2x)^2 + 2(2x)(3) + (3)^2$	Simplify each term
$4x^2 + 12x + 9$	Final answer

Example 6: Find the product

$(2x+y)(2x-y)(4x^2+y^2)$	Multiply first two binomials with difference of squares formula
$(4x^2 - y^2)(4x^2 + y^2)$	Multiply solution by last binomial with difference of squares formula
$16x^4 - y^4$	Final answer

Factoring Polynomials

To factor a polynomial means to do the reverse of multiplying, that is to find an equivalent expression that is a product. Factoring is an important algebraic skill and in this section we study the types of factorization that will commonly arise in your study of mathematics.

Terms with common factors:

When factoring, you should always look for factors common to all the terms of an expression.

Example 7: Factor

$4y^{2n} - 8$ Factor out common factor of 4

$4(y^{2n} - 2)$ Final answer

Example 8: Factor

$5x^{4n} - 20x^{3n}$ Factor out common factor of $5x^{3n}$

$5x^{3n}(x^n - 4)$ Final answer

Example 9: Factor

$8p^{5n}q^{2n} - 4p^{4n}q^{3n} + 2p^{4n}q^{4n}$ Factor out common factor of $2p^{4n}q^{2n}$

$2p^{4n}q^{2n}(4p^n - 2q^n + q^{2n})$ Final answer

Factor by grouping

In more complicated expressions, there may be a common binomial factor

Example 10: Factor

$(a - b)(x + 7) + (a - b)(x - y^2)$ Factor out common binomial factor of $(a - b)$

$(a - b)[(x + 7) + (x - y^2)]$ Combine like terms in second factor

$(a - b)(2x + 7 - y^2)$ Final answer

Example 11: Factor

$\underbrace{ax^2 - a}\ \underbrace{- x^2 + 1}$ Factor GCF out of first and second group

$a(x^2 - 1) - 1(x^2 - 1)$ Factor common binomial factor

$(x^2 - 1)(a - 1)$ Factor difference of squares

$(x + 1)(x - 1)(a - 1)$ Final answer

Factor difference of squares

Example 12: Factor

$25x^{2n} - 49y^{2n}$ Use difference of squares formula

$(5x^n + 7y^n)(5x^n - 7y^n)$ Final answer

Example 13: Factor

$(r-3)^2 - (s-3)^2$ Difference of squares on groups

$[(r-3)+(s-3)][(r-3)-(s-3)]$ Distribute negative through parentheses

$(r-3+s-3)(r-3-s+3)$ Combine like terms in each factor

$(r+s-6)(r-s)$ Final answer

Example 14: Factor

$81x^4 - (y-3z)^2$ Difference of squares formula

$[9x^2 + (y-3z)][9x^2 - (y-3z)]$ Distribute negative through parentheses

$(9x^2 + y - 3z)(9x^2 - y + 3z)$ Final answer

Factor sum and difference of two cubes

Example 15: Factor

$u^3 + 8v^3$ Sum of cubes formula

$(u+2v)(u^2 - 2uv + (2v)^2)$ Simplify

$(u+2v)(u^2 - 2uv + 4v^2)$ Final answer

Example 16: Factor:

$64c^3 - 27d^6$ Identify cube roots

$(4c)^3 - (3d^2)^3$ Difference of cubes formula

$(4c - 3d^2)((4c)^2 + (4c)(3d^2) + (3d)^2)$ Simplify

$(4c - 3d^2)(16c^2 + 12cd^2 + 9d^4)$ Final answer

Factor trinomials

A factorization of a trinomial $px^2 + qx + r$ (also called *quadratic expression*) where p and q are integers, must be of the form

$$px^2 + qx + r = (ax + b)(cx + d)$$

Where $a, b, c,$ and d are integers. It falls that $ac = p$, $bd = r$, and $ad + bc = q$. Only a limited number of choices for $a, b, c,$ and d satisfy these conditions. If none of the choices work, then $px^2 + qx + r$ is irreducible. Trying the various possibilities, as we will see in the next example, is called the method of trial and error. This method is also applicable to trinomials of the form $px^2 + qxy + ry^2$, in which case the factorization must be of the form$(ax + by)(cx + dy)$.

Example 17: Factor

$$6x^2 + 7x - 20$$

Identify relationships of $(ax + b)(cx + d)$

$$ac = 6$$
$$bd = -20$$
$$ad + bc = 7$$

Assume a and c are both positive, consider possible combinations for each

a	1	6	2	3
c	6	1	3	2

This gives the following possibilities

$$6x^2 + 7x - 20 = (x + b)(6x + d)$$
$$6x^2 + 7x - 20 = (6x + b)(x + d)$$
$$6x^2 + 7x - 20 = (3x + b)(2x + d)$$
$$6x^2 + 7x - 20 = (2x + b)(3x + d)$$

Next consider possible values for b and d, as $bd = -20$ we list possible combinations

b	1	−1	2	−2	4	−4	5	−5
d	−20	20	−10	10	−5	5	−4	4

Trying various values we find $b = -4, d = 5$, $a = 3$, and $c = 2$

$$(3x - 4)(2x + 5)$$

Final answer

As a check, you should multiply the final factorization to see whether the original polynomial is obtained.

Factoring with GCF

Example 18: Factor

$-2x^2 - 4x + 2x^3$	Reorder
$2x^3 - 2x^2 - 4x$	Factor GCF of $2x$
$2x(x^2 - x - 2)$	Factor trinomial
$2x(x-2)(x+1)$	Final answer

Example 19: Factor

$2x^2 + 20x + 50$	Factor GCF of 2
$2(x^2 + 10x + 25)$	Note form of perfect square trinomial
$2(x^2 + 2(5)(x) + 5^2\ )$	Perfect square formula
$2(x+5)^2$	Final answer

Example 20: Factor

$63 - 2x^n - x^{2n}$ Using the ac method, split middle term

Product = ac	Sum = b
-63 $(-9)(7) = -63$	-2 $-9 + 7 = -2$

Split middle term using -9 and 7

$63 - 9x^n + 7x^n - x^{2n}$	Factor by grouping
$9(7 - x^n) + x^n(7 - x^n)$	Factor binomial GCF
$(7 - x^n)(9 + x^n)$	Final answer

Solving Quadratic Equations

Definition of a *Quadratic Equation*: A quadratic equation in x is an equation that can be written in the standard form $ax^2 + bx + c = 0$, where $a, b,$ and c are real numbers with $a \neq 0$. A quadratic equation in x is also known as a *second-degree polynomial* in x. We will discuss four methods for solving quadratic equations: factoring, extracting square roots, completing the square, and the quadratic formula.

The first technique is based on the *zero-factor property*.

If $ab = 0$ then $a = 0$ or $b = 0$.

Solving by factoring

Example 21: Solve

$x^2 - 3x - 10 = 0$	Factor
$(x-5)(x+2) = 0$	Set each factor to zero
$x - 5 = 0$ or $x + 2 = 0$	Solve each equation
$x = 5, -2$	Final answer

Example 22: Solve

$2x^2 + 9x + 7 = 3$	Make equal to zero, subtract 3
$2x^2 + 9x + 4 = 0$	Factor
$(2x+1)(x+4) = 0$	Set each factor to zero
$2x + 1 = 0$ or $x + 4 = 0$	Solve each equation
$x = -\frac{1}{2}, -4$	Final answer

Example 23: Solve

$9x^2 - 3x = 0$	Factor GCF
$3x(3x-1) = 0$	Set each factor to zero
$3x = 0$ or $3x - 1 = 0$	Solve each equation
$x = 0, \frac{1}{3}$	Final answer

Example 24: Solve

$49x^2 - 14x + 1 = 0$	Factor
$(7x-1)^2 = 0$	Set factor to zero
$7x - 1 = 0$	Solve the equation
$x = \frac{1}{7}$	Final answer

Extracting square roots

The equation $u^2 = d$, where $d > 0$, has exactly two solutions: $u = \pm\sqrt{d}$.

Example 25: Solve

$4x^2 = 20$	Isolate exponent by dividing by 4
$x^2 = 5$	Square root of both sides
$x = \pm\sqrt{5}$	Final answer

Example 26: Solve

$(x-4)^2 = 7$	Square root of both sides
$x - 4 = \pm\sqrt{7}$	Add 4
$x = 4 \pm \sqrt{7}$	Final answer

Completing the square

Example 27: Solve

$x^2 + 2x - 7 = 0$	Isolate terms with x^2 and x. Make sure coefficient of x^2 is 1
$x^2 + 2x = 7$	Half of coefficient of x term
$\left(\frac{1}{2} \cdot 2\right)^2 = 1^2 = 1$	Add 1 to both sides
$x^2 + 2x + 1 = 7 + 1$	Factor perfect square trinomial
$(x + 1)^2 = 8$	Square root of both sides
$x + 1 = \pm\sqrt{8}$	Subtract 1 from both sides
$x = -1 \pm \sqrt{8}$	Simplify square root
$x = 1 \pm 2\sqrt{2}$	Final answer

The quadratic formula

Example 28: Solve

$2x^2 + 5x - 11 = 0$	Solve using the quadratic formula
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$a = 2, b = 5, c = -11$
$x = \dfrac{-5 \pm \sqrt{5^2 - 4(2)(-11)}}{2(2)}$	Evaluate exponent and multiplication
$x = \dfrac{-5 \pm \sqrt{25 + 88}}{4}$	Add under root
$x = \dfrac{-5 \pm \sqrt{113}}{4}$	Final answer

Equations of quadratic type

Example 29: Solve

$x^4 - 25x^2 + 144 = 0$	Factor trinomial
$(x^2 - 16)(x^2 - 9) = 0$	Differences of squares
$(x+4)(x-4)(x+3)(x-3) = 0$	Set each factor to zero and solve
$x = -4, 4, 3, -3$	Final answer

Example 30: Solve

$3x^{2/3} + 4x^{1/3} - 4 = 0$	Let $A = x^{1/3}$ and $A^2 = x^{2/3}$
$3A^2 + 4A - 4 = 0$	Factor
$(3A - 2)(A + 2) = 0$	Set each factor to zero
$3A - 2 = 0$ or $A + 2 = 0$	Solve each equation
$A = \frac{2}{3}$ or $A = -2$	Replace A with $x^{1/3}$
$x^{1/3} = \frac{2}{3}$ or $x^{1/3} = -2$	Cube both sides
$x = \frac{8}{27}, -8$	Final answer

Example 31: Solve

$\left(\frac{x+1}{x+3}\right)^2 + \left(\frac{x+1}{x+3}\right) = 6$	Let $A = \left(\frac{x+1}{x+3}\right)$ and $A^2 = \left(\frac{x+1}{x+3}\right)^2$
$A^2 + A = 6$	Subtract 6
$A^2 + A - 6 = 0$	Factor
$(A+3)(A-2) = 0$	Set each factor to zero
$A + 3 = 0$ or $A - 2 = 0$	Solve each equation
$A = -3$ or $A = 2$	Replace A with $\left(\frac{x+1}{x+3}\right)$
$\frac{x+1}{x+3} = -3$ or $\frac{x+1}{x+3} = 2$	Multiply by $(x+3)$
$x + 1 = -3(x+3)$ or $x + 1 = 2(x+3)$	Distribute
$x + 1 = -3x - 9$ or $x + 1 = 2x + 6$	Move x terms to left and number to right
$4x = -10$ or $-x = 5$	Divide to isolate x
$x = -\frac{5}{2}$ or $x = -5$	Final answer

1.3 Quadratic Expressions and Equations Practice

Factor

1. $x^{2n+1} + 2x^{n+1}$
2. $x^{2n} - y^{2n}$
3. $(ab)^{2n} - c^{4n}$
4. $a^{2n} - 4$
5. $x^{4n} - y^{2n}$
6. $x^{4n} - 1$
7. $16 - x^{4n}$
8. $y^{a+2} + y^2$
9. $x^4y^2 + 2x^3y + 3x^2y$
10. $2a + ax^n - 3ax^{2n}$
11. $(a-b)(x-8) + (a-b)(x+x^2)$
12. $3x^{3n}y - 18x^{2n}y + 27x^ny$
13. $(a+b)(x-3) + (a+b)(x-4)$
14. $x^2 - (a+1)^2$
15. $x^2 - 1 - x(x-1)$
16. $\pi R^2 - \pi r^2$
17. $n^2 + 2n + np + 2p$
18. $\frac{1}{3}\pi r^2 h + \frac{1}{3}\pi R^2 h - \frac{2}{3}\pi rRh$
19. $2x^2 - 4x + xz - 2z$
20. $45 - 12x^n - x^{2n}$
21. $18m^{4n} + 12m^{3n} + 2m^{2n}$
22. $3a(x-y) + 2b(y^2 - x^2)$
23. $(x-a)^2 - (y-b)^2$
24. $a^3 + a^2 - b^3 - b^2$
25. $(b-4)^2 - a^2$
26. $56 - x^n - x^{2n}$
27. $9a^2 - (a+3b)^2$
28. $15 - x^n - 2x^{2n}$
29. $(s-2)^2 - (t-2)^2$
30. $36 - 3x^n - 3x^{2n}$
31. $a^2y - a - ay^2 + y$
32. $36 - 13x^n + x^{2n}$
33. $10p^6q^2 - 4p^5q^3 - 6p^4q^4$
34. $4x^{4n} - 13x^{2n} + 9$
35. $ax^2 + ay + bx^2 + by$
36. $x^3 - x^2 - x + 1$
37. $5c(a^3 + b^3) - (a^3 + b^3)$
38. $x^5 + 32$
39. $x^2 + x + xy + y$
40. $\frac{3x^2}{a^2} + 7 - \frac{6a^2}{x^2}$

41. $a^2 - 3a + ay - 3y$

42. $a^3 - b^2 + 2ab - b^3 - a^2$

43. $6y^2 - 3y + 2py - p$

44. $a^4 + b^4 - 2a^2b^2 - a^2 + b^2$

45. $(a + b)^2 - c^2$

46. $64a^6 - 729b^6$

47. $ay^2 - a - y^2 + 1$

48. $x^3 + 3x^2y + 3xy^2 + y^3$

49. $15x^2y - 20xy - 35y$

50. $a^4 - (a - 2)^2$

51. $(a - 3)^2 - (b - 3)^2$

52. $x^4 + x^3y - xy^3 - y^4$

53. $c^2xy - c^3 - x^2y + cx$

54. $x(x + 1)(4x - 5) - 6(x + 1)$

55. $xy - x^2y^2 + x^2y - x$

56. $3(2x - y)^2 + 5(2x - y) - 12$

57. $8x^3 - 27y^3$

58. $2(w^2 - 1) - 7(1 - w^2)$

59. $a^4 - 2a^2b^2 + b^4$

60. $(r^2 - 1)^2 - (r - 1)^2$

61. $24x^{2n} - 6x^ny^n - 18y^{2n}$

62. $3x^2 - 27x - (9 - x)^2$

63. $x^2 + 6y - 9 - y^2$

64. $64y^2 - p^2 - 4 - 4p$

65. $a^2 - b^2 - 2b - 1$

66. $2x(1 - x) + 3(x - 1)$

67. $5p^4 - 80$

68. $(a - 2)^2 - 5b(a - 2) - 24b^2$

69. $6x^{2n} + 11x^n - 10$

70. $(a - 3b)^2 - (2b - 3a)^2$

71. $x^2(x + 1) - 4x - 4$

72. $3ax + 6ay - 4ax^2 - 8axy$

73. $4a^2b^2 - (a^2 + b^2 - c^2)^2$

74. $(a + 3b)^3 + (2a + b)^3$

75. $2x^{2n} - 6x^ny^n + 4y^{2n}$

76. $6a^2b - 9ac + 8abc - 12c^2$

77. $x^6 - y^6$

78. $16x^3y - 24x^2y^2 + 6xy - 9y^2$

79. $ac - 6bd + 2ad - 3bc$

80. $x^4 + 4$

81. $20x^{2n} - x^n - 12$

82. $x^2 - y^2 + 6yz - 9z^2$

83. $x^4 - (y^2 - 9)^2$

84. $a^5 + b^5 - a^2b^3 - a^3b^2$

85. $4x^2 - 25y^2 + 2x + 5y$

86. $x^2 + 4y^2 + z^2 + 4xy + 2xz + 4yz$

87. $9x^3(2a+b) - x(2a+b)$

88. $a^2 + 9b^2 + 4c^2 - 6ab + 4ac - 12bc$

89. $2x^{2n} - x^n y^n - 3y^{2n}$

90. $4k^2 - 12kl + 16m^2 - 24lm + 9l^2 + 16km$

91. $2ax^{3n} - 2ax^{2n} - 12ax^n$

92. $x^4 - x^2y^2 + 16y^4$

93. $k^{2n} - 2k^n - 48$

94. $a^2 + 4b^2 + 9c^2 + 4ab - 6ac - 12bc$

95. $3x^{3n} - 3x^{2n}y^n - 6x^n y^{2n}$

96. $4x^4y^2 - 2x^2y + 6x^2y^3$

97. $1 - 6mn - 9n^2 - m^2$

98. $x^4 + 2x^2y^2 + 9y^4$

99. $(a+b+c)^3 - (a^3 + b^3 + c^3)$

100. $(a-b)^3 + (b-c)^3 - (a-c)^3$

Solve each of the following equations. Some equations will have complex roots

101. $x^4 - 5x^2 + 4 = 0$

102. $y^4 - 9y^2 + 20 = 0$

103. $m^4 - 7m^2 - 8 = 0$

104. $y^4 - 29y^2 + 100 = 0$

105. $a^4 - 50a^2 + 49 = 0$

106. $b^4 - 10b^2 + 9 = 0$

107. $x^4 - 25x^2 + 144 = 0$

108. $y^4 - 40y^2 + 144 = 0$

109. $m^4 - 20m^2 + 64 = 0$

110. $x^6 - 35x^3 + 216 = 0$

111. $z^6 - 216 = 19z^3$

112. $y^4 - 2y^2 = 24$

113. $6z^4 - z^2 = 12$

114. $x^{-2} - x^{-1} - 12 = 0$

115. $x^{2/3} - 35 = 2x^{1/3}$

116. $5y^{-2} - 20 = 21y^{-1}$

117. $y^{-6} + 7y^{-3} = 8$

118. $x^4 - 7x^2 + 12 = 0$

119. $x^4 - 2x^2 - 3 = 0$

120. $x^4 + 7x^2 + 10 = 0$

121. $2x^4 - 5x^2 + 2 = 0$

122. $2x^4 - x^2 - 3 = 0$

123. $x^4 - 9x^2 + 8 = 0$

124. $x^6 - 10x^3 + 16 = 0$

125. $8x^6 - 9x^3 + 1 = 0$

126. $8x^6 + 7x^3 - 1 = 0$

127. $x^8 - 17x^4 + 16 = 0$

128. $(x-1)^2 - 4(x-1) = 5$

129. $(y+b)^2-4(y+b)=21$

130. $(x+1)^2+6(x+1)+9=0$

131. $(y+2)^2-6(y+2)=16$

132. $(m-1)^2-5(m-1)=14$

133. $(x-3)^2-2(x-3)=35$

134. $(a+1)^2+2(a+1)=15$

135. $(r-1)^2-8(r-1)=20$

136. $2(x-1)^2-(x-1)=3$

137. $3(y+1)^2-14(y+1)=5$

138. $(x^2-3)^2-2(x^2-3)=3$

139. $(3x^2-2x)^2+5=6(3x^2-2x)$

140. $(x^2+x+3)^2+15=8(x^2+x+3)$

141. $2(3x+1)^{2/3}-5(3x+1)^{1/3}=88$

142. $(x^2+x)^2-8(x^2+x)+12=0$

143. $(x^2+2x)^2-2(x^2+2x)=3$

144. $(2x^2+3x)^2=8(2x^2+3x)+9$

145. $(2x^2-x)^2-4(2x^2-x)+3=0$

146. $(3x^2-4x)^2=3(3x^2-4x)+4$

147. $(y-2)^6-19(y-2)^3=216$

148. $x-\sqrt{x}-30=0$

149. $5x+12=5\sqrt{5x+12}$

150. $x^2-7\sqrt{x^2-4x+11}=4x-23$

151. $\sqrt{m^2+3m-3}=m^2+3m-23$

152. $2r^2+4r-\sqrt{r^2+2r-3}=9$

153. $c^2-8c+\sqrt{c^2-8c+16}+14=0$

154. $x^2-3x-2\sqrt{x^2-3x+7}=8$

155. $\sqrt{x^2+3}-5x^2-9=0$

156. $2y^2-y+\sqrt{2y^2-y-3}=5$

157. $x^2+2x+3-2\sqrt{x^2+2x+6}=0$

158. $3x^2+x+5\sqrt{3x^2+x-1}=25$

159. $\dfrac{5}{2x+1}+\dfrac{12}{(2x+1)^2}=3$

160. $\left(y-\dfrac{6}{y}\right)^2-4y+\dfrac{22}{y}=5$

161. $\left(\dfrac{x^2-8}{2x}\right)^2-4\left(\dfrac{x^2-8}{2x}\right)+3=0$

162. $\dfrac{x^2+12}{x}+\dfrac{56x}{x^2+12}=15$

163. $\left(\dfrac{x^2+8}{x}\right)^2-11\left(\dfrac{x^2+8}{x}\right)+18=0$

164. $\left(\dfrac{x^2-36}{x}\right)^2-4\left(\dfrac{x^2-36}{x}\right)=45$

165. $\dfrac{x}{x-1}-6\sqrt{\dfrac{x}{x-1}}=40$

166. $\dfrac{2x+1}{x}-30=7\sqrt{\dfrac{2x+1}{x}}$

167. $5\left(\frac{x+2}{x-2}\right)^2 = 3\left(\frac{x+2}{x-2}\right) + 2$

168. $\left(\frac{x+1}{x+3}\right)^2 + \left(\frac{x+1}{x+3}\right) = 6$

169. $\left(\frac{x^2+12}{x}\right)^2 - \left(\frac{x^2+12}{x}\right) = 56$

170. $\left(\frac{x+8}{x-8}\right)^2 - 6 = 5\left(\frac{x+8}{x-8}\right)$

171. $\left(\frac{x^2-40}{x}\right)^2 - 9\left(\frac{x^2-40}{x}\right) + 20 = 2$

172. $\left(\frac{x^2-45}{x}\right)^2 - 8\left(\frac{x^2-45}{x}\right) = 48$

173. $2\left(\frac{x^2-3}{x}\right)^2 + 5\left(\frac{x^2-3}{x}\right) + 2 = 0$

174. $\left(\frac{x^2-8}{x}\right)^2 - 9\left(\frac{x^2-8}{x}\right) + 14 = 0$

175. $2\left(\frac{x^2-18}{x}\right)^2 - 7\left(\frac{x^2-18}{x}\right) = -3$

176. $\left(\frac{x^2-20}{x}\right)^2 + 9\left(\frac{x^2-20}{x}\right) + 8 = 0$

177. $\sqrt{x-3} - \sqrt[4]{x-3} = 2$

178. $(x^2-5x-2)^2 - 5(x^2-5x-2) = -4$

179. $x^{1/2} + \frac{1}{x^{1/2}} = \frac{13}{6}$

180. $\left(y+\frac{2}{y}\right)^2 + 3y + \frac{6}{y} = 4$

181. $x^2+3x+1-\sqrt{x^2+3x+1} = 8$

182. $\frac{2x+1}{x} = 3 + 7\sqrt{\frac{2x+1}{x}}$

183. $\frac{x^2-x+2}{x^2+x+2} + \frac{x^2+x+2}{x^2-x+2} = \frac{5}{2}$

184. $\frac{2x^2-4x+6}{x^2-3x+2} + \frac{9x^2-27x+18}{x^2-2x+3} = 9$

1.4 Simplifying Rational Expressions

A rational expression is a quotient $\frac{p}{q}$ of two polynomials p and q. Since division by zero is not allowed, the domain of $\frac{p}{q}$ consists of all real numbers except those that make the denominator zero.

Rational Expressions:

Quotient	Denominator is zero if	Domain
$\frac{2x^2-5x+4}{x^2-25}$	$x=\pm 5$	All $x \neq \pm 5$

To simplify rational expressions we remove the common factors

$$\frac{ad}{bd}=\frac{a}{b}$$

In simple terms: $\frac{\text{Factor numerator}}{\text{Factor denominator}} = \text{Reduce}$

Products and Quotients of Rational Expressions

Example 1: Simplify

$$\frac{x^2-10x+25}{x^2-1}\cdot\frac{2x-2}{x-5}$$ Property of quotients

$$\frac{(x^2-10x+25)(2x-2)}{(x^2-1)(x-5)}$$ Factor all polynomials

$$\frac{(x-5)^2(2)(x-1)}{(x+1)(x-1)(x-5)}$$ Note the domain, denominator can't be zero

$$x \neq \pm 1,5$$ Divide out common factors

$$\frac{2(x-5)}{x+1}$$ Final answer

Example 2: Simplify

$\dfrac{x+3}{2x-3} \div \dfrac{x^2-9}{2x^2-3x}$	Multiply by reciprocal
$\dfrac{x+3}{2x-3} \cdot \dfrac{2x^2-3x}{x^2-9}$	Property of quotients
$\dfrac{(x+3)(2x^2-3x)}{(2x-3)(x^2-9)}$	Factor all polynomials
$\dfrac{(x+3)(x)(2x-3)}{(2x-3)(x+3)(x-3)}$	Note the domain, denominator can't be zero
$x \neq \dfrac{3}{2}, \pm 3$	Divide out common factors
$\dfrac{x}{x-3}$	Final answer

Add and Subtract Rational Expressions

To add or subtract two rational expressions we usually find a common denominator and use the following properties of quotients:

$$\frac{a}{d} + \frac{c}{d} = \frac{a+c}{d} \quad \text{and} \quad \frac{a}{d} - \frac{c}{d} = \frac{a-c}{d}$$

If the denominators of the expression are not the same, we may obtain a common denominator by multiplying the numerator and denominator of each fraction by a suitable expressions. We usually use the least common denominator (LCD) of the two quotients.

Example 3: Add and subtract

$\frac{t}{t+3}+\frac{4t}{t-3}-\frac{18}{t^2-9}$	Factor denominators
$\frac{t}{t+3}+\frac{4t}{t-3}-\frac{18}{(t+3)(t-3)}$	Note LCD
LCD = $(t+3)(t-3)$	Multiply the first numerator and denominator by $(t-3)$ and the second by $(t+3)$
$\frac{(t-3)}{(t-3)}\cdot\frac{t}{t+3}+\frac{(t+3)}{(t+3)}\cdot\frac{4t}{t-3}-\frac{18}{(t+3)(t-3)}$	Multiply/distribute
$\frac{t^2-3t}{(t-3)(t+3)}+\frac{4t^2+12t}{(t-3)(t+3)}-\frac{18}{(t+3)(t-3)}$	Add numerators
$\frac{5t^2+9t-18}{(t+3)(t-3)}$	Factor numerator
$\frac{(5t-6)(t+3)}{(t+3)(t-3)}$	Note the domain, denominator can't be zero
$t\neq\pm3$	Divide out common factor
$\frac{5t-6}{t-3}$	Final answer

Complex Fractions

A complex fraction is a quotient in which the numerator and/or the denominator is a fractional expression.

Example 4: Simplify

$\dfrac{\frac{3}{x+2}-\frac{3}{a+2}}{x-a}$	First subtract numerators by finding a common denominator of $(x+2)(a+2)$
$\dfrac{\left(\frac{3(a+2)-3(x+2)}{(x+2)(a+2)}\right)}{x-a}$	Simplify numerator, distribute
$\dfrac{\frac{3a+6-3x-6}{(x+2)(a+2)}}{x-a}$	Simplify numerator, combine like terms
$\dfrac{\frac{3a-3x}{(x+2)(a+2)}}{x-a}$	Multiply by reciprocal of denominator
$\dfrac{3a-3x}{(x+2)(a+2)}\cdot\dfrac{1}{x-a}$	Property of quotients
$\dfrac{(3a-3x)}{(x+2)(a+2)(x-a)}$	Factor polynomials
$\dfrac{3(a-x)}{(x+2)(a+2)(x-a)}$	Note the domain, denominator can't be zero
$x\neq -2, a$	Replace $\frac{a-x}{x-a}$ with -1
$\dfrac{-3}{(x+2)(a+2)}$	Final answer

An alternate method is to multiply the numerator and denominators of the given expression by $(x+2)(a+2)$, the LCD of the numerator and denominator, and then simplify the result.

Example 5: Simplify if $h \neq 0$

$$\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$ Multiply by LCD of $x^2(x+h)^2$

$$\frac{\frac{x^2(x+h)^2}{1} \cdot \frac{1}{(x+h)^2} - \frac{1}{x^2} \cdot \frac{x^2(x+h)^2}{1}}{h \cdot x^2(x+h)^2}$$ Reduce denominators

$$\frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$ Square $(x+h)$

$$\frac{x^2 - (x^2 + 2hx + h^2)}{hx^2(x+h)^2}$$ Distribute negative

$$\frac{x^2 - x^2 - 2hx - h^2}{hx^2(x+h)^2}$$ Combine like terms

$$\frac{-2hx - h^2}{hx^2(x+h)^2}$$ Factor numerator (GCF of h)

$$\frac{h(-2x - h)}{hx^2(x+h)^2}$$ Divide out h

$$\frac{-2x - h}{x^2(x+h)^2}$$ Final answer

Some problems have multiple ways they can be simplified, as shown in examples 6, 7, and 8.

Example 6: Simplify by using definition of negative exponents

$$\frac{(1-x^2)^{1/2}(2x)-x^2\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)}{[(1-x^2)^{1/2}]^2}$$

Definition of a negative exponent and multiply $x^2\left(\frac{1}{2}\right)$ by $-2x$ and multiplying exponents in denominator

$$\frac{(1-x^2)^{1/2}(2x)+\frac{x^3}{(1-x^2)^{1/2}}}{1-x^2}$$

Getting a common denominator in the numerator, multiplying the first fraction by $(1-x^2)^{1/2}$

$$\frac{\frac{2x(1-x^2)}{(1-x^2)^{1/2}}+\frac{x^3}{(1-x^2)^{1/2}}}{1-x^2}$$

Multiply numerator

$$\frac{\frac{2x-2x^3}{(1-x^2)^{1/2}}+\frac{x^3}{(1-x^2)^{1/2}}}{1-x^2}$$

Add numerators

$$\frac{\frac{-x^3+2x}{(1-x^2)^{1/2}}}{1-x^2}$$

Multiply by reciprocal

$$\frac{-x^3+2x}{(1-x^2)^{1/2}}\cdot\frac{1}{1-x^2}$$

Multiply

$$\frac{-x^3+2x}{(1-x^2)^{3/2}}$$

Final answer

Example 7: Simplify by eliminating negative exponents

$$\frac{(1-x^2)^{1/2}(2x) - x^2\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)}{[(1-x^2)^{1/2}]^2}$$

Eliminate negative powers by multiplying by $(1-x^2)^{1/2}$ and multiply $x^2\left(\frac{1}{2}\right)$ by $-2x$ and multiplying exponents in denominator

$$\frac{(1-x^2)^{1/2}(2x) + x^3(1-x^2)^{-1/2}}{(1-x^2)} \cdot \frac{(1-x^2)^{1/2}}{(1-x^2)^{1/2}}$$

Simplify

$$\frac{2x(1-x^2) + x^3}{(1-x^2)^{3/2}}$$

Distribute in numerator

$$\frac{2x - 2x^3 + x^3}{(1-x^2)^{3/2}}$$

Combine like terms in numerator

$$\frac{-x^3 + 2x}{(1-x^2)^{3/2}}$$

Final answer

Example 8: Simplify by factoring the GCF first

$$\frac{(1-x^2)^{1/2}(2x) - x^2\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)}{[(1-x^2)^{1/2}]^2}$$

Factor GCF of $x(1-x^2)^{-1/2}$ and multiply $\frac{1}{2}$ by -2 and multiply exponents in denominator

$$\frac{x(1-x^2)^{-1/2}[2(1-x^2) + x^2]}{1-x^2}$$

Distribute 2 in numerator and move negative exponent to denominator

$$\frac{x(2 - 2x^2 + x^2)}{(1-x^2)^{3/2}}$$

Combine like terms in numerator

$$\frac{x(-x^2 + 2)}{(1-x^2)^{3/2}}$$

Distribute x in numerator

$$\frac{-x^3 + 2x}{(1-x^2)^{3/2}}$$

Final answer

1.4 Simplifying Rational Expressions Practice

Simplify each of the following fractional expressions

1. $\dfrac{x^2-a^2}{(x+a)^2}\cdot\dfrac{2x+2a}{3x}$

2. $\dfrac{4m^2-1}{m^2-16}\cdot\dfrac{m+4}{2m+1}$

3. $\dfrac{c^4-d^4}{(c-d)^4}\div\dfrac{c^2+d^2}{c-d}$

4. $\dfrac{xy^2-y^3}{x^3+x^2y}\cdot\dfrac{x^2-xy-2y^2}{x^2-2xy+y^2}$

5. $\dfrac{3t}{y^2-6y+8}\div\dfrac{2t}{y^2-y-12}$

6. $\dfrac{r^3-s^3}{r-s}\div\dfrac{2r^2+2rs+2s^2}{2r+2s}$

7. $\dfrac{x^2-1}{2x-4}\cdot\dfrac{x^2-4}{x^2-x-2}\div\dfrac{x^2+x-2}{3x-6}$

8. $\dfrac{m^2-1}{16m^2-9n^2}\cdot\dfrac{4m-3n}{2m^2+1}\div\dfrac{m-1}{4m+3n}$

9. $\dfrac{\frac{x}{y}+\frac{y}{x}}{x^4-y^4}$

10. $\dfrac{x-y}{\frac{x}{y}-\frac{y}{x}}$

11. $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x^2}}$

12. $\dfrac{a^2-\frac{a}{b}}{b-\frac{1}{a}}$

13. $\dfrac{\frac{x}{1+x}+\frac{1-x}{x}}{\frac{x}{1+x}-\frac{1-x}{x}}$

14. $\dfrac{\frac{a}{a-b}-1}{\frac{a}{a+b}-1}$

15. $\dfrac{1-\frac{a}{a-b}}{1+\frac{a}{a-b}}$

16. $\dfrac{x^{-2}-y^{-2}}{x^{-1}+y^{-1}}$

17. $\dfrac{x^{-2}y+xy^{-2}}{x^{-2}y^{-2}}$

18. $\dfrac{x^{-2}y+xy^{-2}}{x^{-2}-y^{-2}}$

19. $\dfrac{x^{-1}y+2+xy^{-1}}{x+y}$

20. $\dfrac{x^{-3}y-xy^{-3}}{x^{-2}-y^{-2}}$

21. $\dfrac{x^{-2}+x^{-1}-2}{x^{-1}-1}$

22. $\dfrac{(x+1)^{-1}-(x+1)}{x}$

23. $\dfrac{(x-1)^{-2}-1}{(x-1)^{-2}+1}$

24. $\dfrac{(x+1)^{-1}-(x-1)^{-1}}{(x+1)^{-1}+(x-1)^{-1}}$

25. $\dfrac{(x+y)^{-1}+(x+y)^{-2}}{(x+y)^{-2}-(x+y)^{-1}}$

26. $\dfrac{4-4x^{-1}+x^{-2}}{4-x^{-2}}$

27. $\dfrac{x^{-2}-6x^{-1}+9}{x^{-2}-9}$

28. $\dfrac{x^{-3}+y^{-3}}{x^{-2}-x^{-1}y^{-1}+y^{-2}}$

29. $\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\div\left(\dfrac{x}{y}-\dfrac{y}{x}\right)$

30. $\left(\dfrac{y}{x-y}-1\right)\div\left(\dfrac{x}{y-x}\right)$

31. $\left(\dfrac{m}{3}-\dfrac{3}{m}\right)\div\left(\dfrac{2m-6}{3m}\right)$

32. $\left(\dfrac{x^2}{y}-y\right)\div\left(\dfrac{x}{y}-1\right)$

33. $\left(1-\dfrac{x}{x+y}\right)\div\left(\dfrac{y}{x+y}\right)$

34. $\left(\dfrac{1}{a+x}-\dfrac{1}{x-a}\right)\div\left(\dfrac{1}{a-x}-\dfrac{1}{x+a}\right)$

35. $\left(4-\dfrac{6}{a+1}\right)\div\left(8-\dfrac{4a-8}{a^2-1}\right)$

36. $\left(\dfrac{2x}{x-2}-\dfrac{x}{x-1}\right)\div\left(\dfrac{3x}{x-3}+\dfrac{2x}{2-x}\right)$

37. $\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right)\div\left(\dfrac{a}{a+b}-\dfrac{b}{a-b}\right)$

38. $\dfrac{1}{x}\left(x-\dfrac{1}{x}\right)\div\dfrac{x^2-1}{x^2}$

39. $\left(a+\dfrac{1}{b}\right)\left(b-\dfrac{1}{a}\right)\div\dfrac{a^2b^2-1}{3ab}$

40. $\left(x-\dfrac{9}{x}\right)\left(\dfrac{6}{x^2}+\dfrac{1}{x}-1\right)\left(\dfrac{2-x^2}{3x-x^2}+1\right)$

41. $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{(x-h)^2}}{h}$

42. $x\left(\dfrac{x}{y}-\dfrac{y}{x}\right)\div\left(1-\left[\dfrac{x}{y}\right]^2\right)$

43. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a^2+x^2}{a^2-x^2}}{\dfrac{a+x}{x-a}+\dfrac{a^2+x^2}{a^2-x^2}}$

44. $\dfrac{x^2}{\sqrt{x^2+1}}-\sqrt{x^2+1}$

45. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

46. $\dfrac{\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}}{h}$

47. $\dfrac{\sqrt{x}-\dfrac{1}{2\sqrt{x}}}{\sqrt{x}}$

48. $\dfrac{3x^{1/3}-x^{-2/3}}{3x^{-2/3}}$

49. $\dfrac{\dfrac{t^2}{\sqrt{t^2+1}}-\sqrt{t^2+1}}{t^2}$

50. $\dfrac{-x^3(1-x^2)^{-1/2}-2x(1-x^2)^{1/2}}{x^4}$

51. $$\frac{x(x+1)^{-3/4}-(x+1)^{1/4}}{x^2}$$

52. $$\frac{(2x+1)^{1/3}-\dfrac{4x}{3(2x+1)^{2/3}}}{(2x+1)^{3/2}}$$

53. $$\frac{\sqrt{2x-1}-\dfrac{x+2}{\sqrt{2x-1}}}{2x-1}$$

54. $\dfrac{y+1}{y-1}$ where $y=\dfrac{x}{x-1}$

55. $\dfrac{y^2-y+1}{y+1}$ where $y=1+\dfrac{1}{x}$

56. $\dfrac{x-a}{x}\sqrt{x^2-4}$ where $x=a+\dfrac{1}{a}$

57. a^2+ax where $x=a+\dfrac{1}{a}$ and $a=\dfrac{1}{\sqrt{2}-1}$

58. $$\frac{a^3+b^3}{a^2+3ab+2b^2}\cdot\frac{3a-6b}{3a^2-3ab+3b^2}\div\frac{a^2-4b^2}{a+2b}$$

59. $$\frac{a^2+7ab+10b^2}{a^2+6ab+5b^2}\cdot\frac{a+b}{a^2+4ab+4b^2}\div\frac{1}{a+2b}$$

60. $$\frac{x^{2n}+3x^n+9}{x^{2n}+x^n-12}\cdot\frac{x^{2n}+2x^n-8}{x^{3n}-27}\div\frac{x^{2n}-4}{x^{2n}-6x^n+9}$$

61. $$\frac{2x^{2n}+7x^n-15}{2x^{2n}-3x^n-14}\cdot\frac{2x^{2n}-19x^n+42}{2x^n-3}\div\frac{x^{2n}-x^n-30}{x^n+2}$$

62. $$\frac{x^{2a}+3x^a-10}{x^{2a}+6x^a+5}\cdot\frac{2x^{2a}-x^a-3}{x^{2a}+x^a-6}\div\frac{8x^a+20}{6x^a+15}$$

63. $$\left(x-\frac{9}{x}\right)\left(\frac{2-x^2}{3x-x^2}+1\right)\div\left(\frac{6}{x^2}+\frac{1}{x}-1\right)$$

64. $$\frac{-4x(x^2-3)^{-3}(6x+1)^3-18(6x+1)^2(x^2-3)^{-2}}{(6x+1)^6}$$

65. $$\frac{-6x(3x+2)^5(x^2+1)^{-4}-15(3x+2)^4(x^2+1)^{-3}}{(3x+2)^{10}}$$

66. $$\frac{(6x+1)^3(27x^2+2)-16(27x^3+2x)(6x+1)^2}{(6x+1)^3(27x^2+2)}$$

67. $$\frac{2(4x^2+9)^{1/2}-4x(2x+3)(4x^2+9)^{-1/2}}{4x^2+9}$$

68. $$\frac{2(3x+2)^{3/4}(2x+3)^{-2/3} - 3(3x+2)^{-1/4}(2x+3)^{1/3}}{(3x+2)^{3/2}}$$

69. $$\frac{2(3x-1)^{1/3} - (2x+1)(3x-1)^{-2/3}}{(3x-1)^{2/3}}$$

70. $$\frac{\frac{1}{2}(x+1)(2x-3x^2)^{-1/2}(2-6x) - (2x-3x^2)^{1/2}}{(x+1)^2}$$

1.5 Complex Numbers

In the real number system, negative numbers do not have square roots. Mathematicians have invented a larger number system that contain the real number system but is such that negative numbers do have square roots. That system is called the complex number system and makes use of the number i.

We define the number i so that $i^2 = -1$. Thus $i = \sqrt{-1}$.

Example 1: Express in terms of i

$\sqrt{-7}$	Factor out -1
$\sqrt{(-1)(7)}$	Make two radicals
$\sqrt{-1}\sqrt{7}$	Use definition: $\sqrt{-1} = i$
$i\sqrt{7}$	Final answer

Example 2: Express in terms of i

$\sqrt{-4}$	Factor out -1
$\sqrt{(-1)(4)}$	Make two radicals
$\sqrt{-1}\sqrt{4}$	Take roots
$i \cdot 2$	Rewrite
$2i$	Final answer

Example 3: Express in terms of i

$-\sqrt{-11}$	Factor out -1
$-\sqrt{(-1)(11)}$	Make two radicals
$-\sqrt{-1}\sqrt{11}$	Take root
$-i\sqrt{11}$	Final answer

Example 4: Express in terms of i

$-\sqrt{-64}$	Factor out -1
$-\sqrt{(-1)(64)}$	Make two radicals
$-\sqrt{-1}\sqrt{64}$	Take roots
$-i \cdot 8$	Rewrite
$-8i$	Final answer

Example 5: Express in terms of i

$\sqrt{-32}$	Factor
$\sqrt{(-1)(16)(2)}$	Make three radicals
$\sqrt{(-1)}\sqrt{16}\sqrt{2}$	Take roots
$i \cdot 4\sqrt{2}$	Rewrite
$4i\sqrt{2}$	Final answer

An imaginary number is a number that can be written bi where b is some real number and $b \neq 0$.

To form the system of complex numbers, we take the imaginary numbers and the real numbers, as well as all possible sums of real and imaginary numbers. These are complex numbers:

$$4 - 7i, \qquad -\pi + 11i, \qquad 47, \qquad i\sqrt{7}$$

A complex number is any number that can be written $a + bi$, where a and b are real numbers. (Note that a and/or b can be 0)

Addition and Subtraction

Complex numbers follow the commutative and associative laws of addition. Thus we can add and subtract them as we do binomials in real numbers.

Example 6: Add

$(7+5i)+(3+4i)$ Collect real and imaginary parts

$10+9i$ Final answer

Example 7: Subtract

$(4+3i)-(2-2i)$ Collect real and imaginary parts

$(4-2)+\big(3i-(-2i)\big)$ Note that both 2 and the $-2i$ are being subtracted

$2+5i$ Final answer

Multiplying

Complex numbers also obey the commutative and associative laws of multiplication. However, the property $\sqrt{a}\sqrt{b}=\sqrt{ab}$ does not hold for imaginary numbers. That is, all square roots of negatives must be expressed in terms of i before we multiply. For example:

Correct: $\sqrt{-2}\sqrt{-5}=i\sqrt{2}\cdot i\sqrt{5}=i^2\sqrt{10}=-\sqrt{10}$

Incorrect: $\sqrt{-2}\sqrt{-5}=\sqrt{(-2)(-5)}=\sqrt{10}$ (Note, this is incorrect, missing the negative)

Example 8: Multiply

$\sqrt{-9}\sqrt{-49}$ Express in terms of i

$i\sqrt{9}\cdot i\sqrt{49}$ Take square roots

$i\cdot 3\cdot i\cdot 7$ Multiply

$21i^2$ Use definition, $i^2=-1$

$21(-1)$ Multiply

-21 Final answer

Example 9: Multiply

$\sqrt{-3}\sqrt{-5}$	Express in terms of i
$i\sqrt{3} \cdot i\sqrt{5}$	Multiply
$i^2\sqrt{15}$	Use definition, $i^2 = -1$
$-\sqrt{15}$	Final answer

Example 10: Multiply

$-2i \cdot 8i$	Multiply
$-16i^2$	Use definition, $i^2 = -1$
$-16(-1)$	Multiply
16	Final answer

Example 11: Multiply

$-3i(4 - 5i)$	Distribute
$-12i + 15i^2$	Use definition, $i^2 = -1$
$-12i - 15$	Rewrite in form $a + bi$
$-15 - 12i$	Final answer

Example 12: Multiply

$(2 + i)(1 + 3i)$	FOIL
$2 + 6i + i + 3i^2$	Use definition, $i^2 = -1$
$2 + 6i + i - 3$	Combine like terms
$-1 + 7i$	Final answer

Complex Conjugates and Division

Consider the following multiplication:

$$(2 - 3i)(2 + 3i) = 4 + 6i - 6i - 9i^2 = 4 + 9 = 13$$

Note that the imaginary terms $6i$ and $-6i$ added to 0, so our answer was a real number. This will happen any time numbers of the form $a + bi$ and $a - bi$ are multiplied. Pairs of numbers like $2 - 3i$ and $2 + 3i$ are known as complex conjugates, or simply, conjugates.

The complex conjugate of a complex number $a + bi$ is $a - bi$

The complex conjugate of a complex number $a - bi$ is $a + bi$

Example 13: Multiply

$(7 + 5i)(7 - 5i)$	Recognize conjugates, imaginary terms in middle will add to zero.
$7^2 - (5i)^2$	Square
$49 - 25i^2$	Use definition, $i^2 = -1$
$49 + 25$	Add
74	Final answer

We use conjugates in dividing complex numbers.

Example 14: Divide

$\dfrac{-3 + 7i}{1 - 4i}$	Multiply by conjugate $1 + 4i$
$\dfrac{-3 + 7i}{1 - 4i} \cdot \dfrac{1 + 4i}{1 + 4i}$	FOIL numerator and denominator
$\dfrac{-3 - 12i + 7i + 28i^2}{1 - 16i^2}$	Use definition, $i^2 = -1$
$\dfrac{-3 - 12i + 7i - 28}{1 + 16}$	Combine like terms
$\dfrac{-31 - 5i}{17}$	Write in form $a + bi$
$-\dfrac{31}{17} - \dfrac{5}{17}i$	Final answer

Note the similarity between this example and rationalizing denominators. The symbol for the number 1 was formed using the conjugate of the divisor.

Example 15: Divide

$\dfrac{4+3i}{5+4i}$	Multiply by the conjugate, $(5-4i)$
$\dfrac{4+3i}{5+4i}\cdot\dfrac{5-4i}{5-4i}$	FOIL numerator and denominator
$\dfrac{20-16i+15i-12i^2}{25-16i^2}$	Use definition, $i^2=-1$
$\dfrac{20-16i+15i+12}{25+16}$	Combine like terms
$\dfrac{32-i}{41}$	Write in form $a+bi$
$\dfrac{32}{41}-\dfrac{1}{41}i$	Final answer

1.5 Complex Numbers Practice

Simplify the following expressions.

1. $(3+2i)+(4-5i)$
2. $(-7+9i)+(2-10i)$
3. $(5-6i)-(3-2i)$
4. $(8+5i)-(2+3i)$
5. $(2+3i\sqrt{7})+(4-5i\sqrt{7})$
6. $(4-2i\sqrt{5})(3+6i\sqrt{5})$
7. $(3-2i)^2$
8. $(4+i)^2$
9. $(2+i\sqrt{3})^2$
10. $(5+2i\sqrt{2})^2$
11. $(2+4i)(3-5i)$
12. $(1+2i)(6+3i)$
13. $(4-5i)(2-i)$
14. $(3+i\sqrt{2})(4+3i\sqrt{2})$
15. $(7+i)(7-i)$
16. $(3+4i)(3-4i)$
17. $(6+2i)(6-2i)$
18. $\dfrac{6}{1-i}$
19. $\dfrac{1+i}{1-i}$
20. $\dfrac{4+3i}{2-3i}$
21. $\dfrac{6+i}{6-i}$
22. $\dfrac{1-4i}{2+4i}$
23. $\dfrac{5+i\sqrt{3}}{5-i\sqrt{3}}$
24. $\dfrac{2+i\sqrt{5}}{1-i\sqrt{5}}$
25. $(1-2i)^3$
26. $\left(\dfrac{1+i}{1-i}\right)^2$
27. $\dfrac{1}{(1+i)^2}$
28. Show that $1+i$ is a solution of the equation $x^2-2x+2=0$
29. Show that $2-3i$ is a solution of the equation $x^2-4x+13=0$
30. Show that $\frac{1}{\sqrt{2}}(1+i)=\sqrt{i}$

1.6 Complete the Square

Equation of the type $ax^2 + bx + c = 0$

When none of the constants $a, b,$ or c is zero, we can try factoring. Unfortunately, many quadratic equations are extremely difficult to solve by factoring. The procedure used in the next example enables us to solve an equation for which factoring would not work.

Example 1: Solve

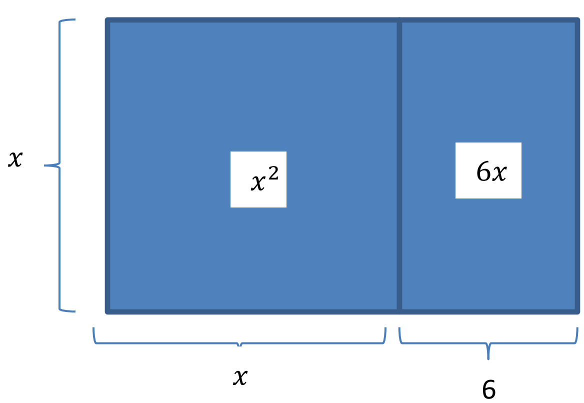
$x^2 + 6x + 4 = 0$	Add -4 on both sides
$x^2 + 6x = -4$	Add 9 to both sides (we explain this shortly)
$x^2 + 6x + 9 = -4 + 9$	Factoring the trinomial square
$(x + 3)^2 = 5$	Square root of both sides (plus or minus!)
$x + 3 = \pm\sqrt{5}$	Add -3 on both sides
$x = -3 \pm \sqrt{5}$	Final answer

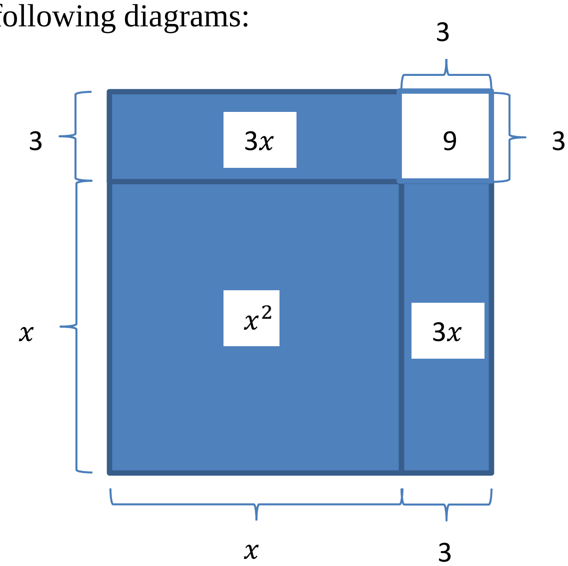
Complete the Square

The decision to add 9 on both sides in example 1 was not made arbitrarily. We choose 9 because it made the left side a trinomial square. The 9 was obtained by taking half of the coefficient of x and squaring it, that is:

$$\left(\frac{1}{2} \cdot 6\right)^2 = 3^2 = 9$$

To help see why this procedure works examine the following diagrams:





Note that both figures represent the same area as $x^2 + 6x$. However, only the figure on the right can be converted into a square with the addition of a constant term. The constant term 9 can be interpreted as the area of the *missing* piece of the diagram on the right. It completes the square.

Example 2: Complete the square

$x^2 - 7x$	Half of the x coefficient squared
$\left(\frac{1}{2} \cdot -7\right)^2 = \left(-\frac{7}{2}\right)^2 = \frac{49}{4}$	Add and subtract (no equals sign so we must balance to zero)
$x^2 - 7x + \frac{49}{4} - \frac{49}{4}$	Factor perfect square trinomial
$\left(x - \frac{7}{2}\right)^2 - \frac{49}{4}$	Final answer

Example 3: Complete the square

$x^2 + \frac{3}{2}x$	Half of the x coefficient squared
$\left(\frac{1}{2} \cdot \frac{3}{2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$	Add and subtract
$x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}$	Factor perfect square trinomial
$\left(x + \frac{3}{4}\right)^2 - \frac{9}{16}$	Final answer

Quadratic Portion

When graphing an expression in future sections, it will be helpful to identify a quadratic portion of an expression and complete the square.

Should x^2 have a coefficient we must first factor it out of the x^2 and x terms, even if it means creating a fraction on the x term. When we complete the square and add and subtract, be sure to multiply by the coefficient on the subtraction to remain balanced to zero.

Example 4: Complete the square on the quadratic portion of the expression

$\dfrac{5x}{4x^2+3x+1}$	Note the quadratic portion in the denominator
$4x^2+3x+1$ $4\left(x^2+\dfrac{3}{4}x\right)+1$	Factor 4 out of x^2 and x terms Half the x coefficient squared
$\left(\dfrac{1}{2}\cdot\dfrac{3}{4}\right)^2=\left(\dfrac{3}{8}\right)^2=\dfrac{9}{64}$	Add inside parenthesis, subtract at end, multiplying by 4
$4\left(x^2+\dfrac{3}{4}x+\dfrac{9}{64}\right)+1-\dfrac{9}{64}(4)$	Factor parenthesis, combine terms at end
$4\left(x+\dfrac{3}{8}\right)^2+\dfrac{7}{16}$	Replace in original expression
$\dfrac{5x}{4\left(x+\frac{3}{8}\right)^2+\frac{7}{16}}$	Final answer

1.6 Complete the Square Practice

Complete the square on the quadratic portion of each of the following expressions

1. $\dfrac{1}{x^2 - 4x - 12}$

2. $\dfrac{4}{4x^2 - 4x - 3}$

3. $\sqrt{x^2 + 2x - 3}$

4. $\sqrt{1 - 2x + 2x^2}$

5. $\dfrac{1}{\sqrt{6x - x^2}}$

6. $\dfrac{1}{\sqrt{16 - 6x - x^2}}$

7. $\sqrt[3]{3x^2 - 9x + 1}$

8. $\dfrac{2x - 1}{\sqrt{2x^2 - 3x + 1}}$

9. $\dfrac{6}{2x^2 - 4x - 2}$

10. $2\sqrt{4x^2 + 2x - 2}$

11. $\dfrac{3x - 1}{9x^2 - 6x + 4}$

12. $\sqrt{9 + 4x - 6x^2}$

13. $\dfrac{x}{1 - 3x - 6x^2}$

14. $\left(\dfrac{1}{2}x^2 + 3x - 2\right)^2$

15. $\dfrac{2}{x^4 - 16x^2}$

16. $\dfrac{6}{4x^6 + 8x^3 + 5}$

1.7 Solving Linear Formulas

A formula is a kind of recipe, or rule, for doing a certain kind of calculation and is often stated in the form of an equation. For example, $P = 4S$ where P represents the perimeter of a square and S the length of a side. Other formulas that you may recall are $A = \pi r^2$ (for the area A of a circle of radius r), $C = \pi d$ (for the circumference C of a circle of diameter d), and $A = bh$ (for the area A of a parallelogram of height h and base length b).

Solving Formulas

The formula $A = lw$ gives the formula for the area of a rectangle A in terms of its length l and its width w.

Example 1: Solve for l

$A = lw$ We want the l alone, so we divide by w

$\frac{A}{w} = l$ Final answer

The formula $I = Prt$ is used to determine the amount of simple interest I, earned on P dollars, when investede for t years at an interest rate r.

Example 2: Solve for r

$I = Prt$ We want the r alone, so we divide by Pt

$\frac{I}{Pt} = r$ Final answer

The formula $A = P + Prt$ tells how much a principal P, in dollars, will be worth when invested at a simple interest rate r in t years.

Example 3: Solve for P

$A = P + Prt$ We need a single P to solve for, factor it out

$A = P(1 + rt)$ Divide out $(1 + rt)$

$\frac{A}{1 + rt} = P$ Final answer

A trapezoid is a geometric shape with four sides, two of which, the bases, are parallel to each other. The formula for calculating the area A of a trapezoid with based b_1 and b_2 and height h is $A = \frac{h}{2}(b_1 + b_2)$.

Example 4: Solve for b_2

$A = \frac{h}{2}(b_1 + b_2)$	Clear fraction by multiplying by 2
$2A = h(b_1 + b_2)$	Divide by h
$\frac{2A}{h} = b_1 + b_2$	Subtract b_1
$\frac{2A}{h} - b_1 = b_2$	Final answer.

To solve a formula for a given letter, identify the letter and:

1. Multiply on both sides to clear fractions or decimals, or to remove grouping symbols if that is needed
2. Get all terms with the letter for which we are solving on one side of the equation and all other terms on the other side, using the addition principle.
3. Collect like terms on each side where convenient. This may require factoring.
4. Solve for the letter in question, using the multiplication principle.

1.7 Solving Linear Formulas Practice

Solve each of the following formulas for the indicated unknown

1. $ax + y = a$, for x
2. $2x - a = 4a - 3$, for x
3. $2p + q = 2q - p$, for q
4. $3(x - a) + 2a = 4x + 5a$, for x
5. $2x + y = xy$, for x
6. $-2(x - b) + 2a = 3b - a$, for x
7. $ab - x = bx$, for x
8. $2x + b = 3x - b$, for x
9. $a + 2b = 3(b - 2a)$, for a, for b
10. $3(x + a) = x - a + 2b$, for x
11. $x + y = x - y$, for y
12. $2a(x + y) = ax$, for x
13. $2y - ax + a = ax - a$, for y
14. $2(x + y) = x + y + 4$, for x
15. $a(m - 2n) + an = 2am$, for m
16. $a(x + a) = x + a^2 + 2a$, for x
17. $mx = nx - 2n$, for x
18. $2(m + n) = 3(m - n) + mn$, for n, m
19. $a(x + y) = 2ax + 2y$, for x, y
20. $2px + pq = qx - 2p$, for x, p, and q
21. $(a + x)(x - y) = 2axy$, for y
22. $Q_1 = P(Q_2 - Q_1)$, for Q_1
23. $L = \pi(r_1 + r_2) + 2d$, for r_1
24. $R = \dfrac{kA(T_1 + T_2)}{d}$, for T_1
25. $P = \dfrac{V_1(V_2 - V_1)}{g^j}$, for V_2
26. $a(m - n) = amn + m - n$, for m, n
27. $x^2 - 2y = 3x^2 + 2y$, for x^2, y
28. $S = \pi r\sqrt{r^2 + h^2}$ for h
29. $i = \pm\sqrt{\dfrac{1}{LC}}\sqrt{Q^2 - q}$, for Q
30. $S = \dfrac{n}{2}[2a + (n - 1)d]$, for d
31. $nE = I(R + nr)$, for r
32. $I = \dfrac{E}{r + \frac{R}{n}}$, for n
33. $\dfrac{v^2}{2g} + \dfrac{P}{c} = H$, for c
34. $T = T_1\left(1 - \dfrac{n - 1}{n} \cdot \dfrac{h}{h_0}\right)$, for h

35. $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$, for C

36. $V = \frac{q}{\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$, for r_1

37. $x - 2a + 3(y + 2b) = x + y + a + b$, for y

38. $r - 3(s - r) + 2(s + r) = 4(r - s)$, for r

39. $x(k + y) - k(x - y) = kx$, for x, y, and k

40. $2(x - y) = 3(x - y) + x - y$, for $(x - y)$

41. $5(2x + y) = 4(2x + y - 1)$, for $(2x + y)$

1.8 Solving Absolute Value Equations and Inequalities

Definition of Absolute Value

The absolute value of a real number a, denoted by $|a|$, is defined as follows:

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

Solving an equation containing an absolute value

To solve an equation with an absolute value, we must consider both the positive and negative options.

Example 1: Solve

$\lvert x - 7\rvert = 5$	Inside absolute value could be positive or negative 5
$x - 7 = 5$ or $x - 7 = -5$	Solve both equations
$x = 12, 2$	Final answer

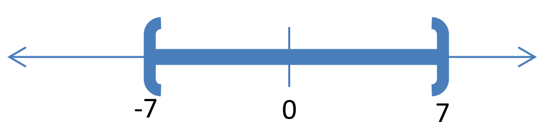
Before removing the absolute value, we must first isolate the absolute value on one side of the equation.

Example 2: Solve

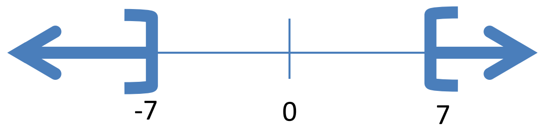
$2\lvert x - 4\rvert + 3 = 17$	Subtract 3 from both sides
$2\lvert x - 4\rvert = 14$	Divide both sides by 2
$\lvert x - 4\rvert = 7$	Consider positive or negative 7
$x - 4 = 7$ or $x - 4 = -7$	Solve both equations
$x = 11, -3$	Final answer

Inequalities Involving Absolute Value

$|x|$ represents the distance along the number line from x to the origin. Thus $|x| < 7$ means that the distance from x to the origin is less than 7.

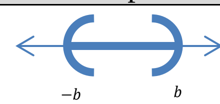
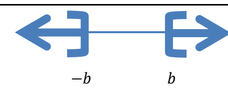


We can see in the figure that this is the set of real numbers x such that $-7 < x < 7$. On the other hand, $|x| \geq 7$ means the distance from x to the origin is greater than or equal to 7.



Therefore the figure shows either $x \geq 7$ or $x \leq -7$.

Properties of Absolute Value

Ab. Value	Equivalent	Interval	Graph	Description
$\lvert a\rvert < b$	$-b < a < b$	$(-b, b)$		Trapped
$\lvert a\rvert \geq b$	$a \leq -b$ or $a \geq b$	$(-\infty, -b] \cup [b, \infty)$		Tails

Example 3: Solve, graph, and give interval notation

$\lvert 5x - 3\rvert < 7$	Write equivalent inequality
$-7 < 5x - 3 < 7$	Add 3 to all three parts
$-4 < 5x < 10$	Divide all three parts by 5
$-\frac{4}{5} < x < 2$	Graph
	Give interval notation
$\left(-\frac{4}{5}, 2\right)$	Final answer

Example 4: Solve, graph, and give interval notation

$\left\|3-\frac{1}{2}x\right\| \geq 12$	Write equivalent inequality
$3-\frac{1}{2}x \leq -12$ or $3-\frac{1}{2}x \geq 12$	Add -3 to each inequality
$-\frac{1}{2}x \leq -15$ or $-\frac{1}{2}x \geq 9$	Multiply each inequality by -2 and change the inequality symbol
$x \geq 30$ or $x \leq -18$	Graph
	Give interval notation
$(-\infty,-18] \cup [30,\infty)$	Final answer

Just as with equations, before writing our equivalent expression, we must first isolate the absolute value

Example 5: Solve and give interval notation

$2\|-11-7x\|-2>10$	Add 2 to both sides
$2\|-11-7x\|>12$	Divide both sides by 2
$\|-11-7x\|>6$	Write equivalent inequality
$-11-7x<-6$ or $-11-7x>6$	Add 11 to each inequality
$-7x<5$ or $-7x>17$	Divide by -7 and change the inequality symbol
$x>-\frac{5}{7}$ or $x<-\frac{17}{7}$	Give interval notation (graph first if it helps)
$\left(-\infty,-\frac{17}{7}\right) \cup \left(-\frac{5}{7},\infty\right)$	Final answer

Example 6: Solve and give interval notation

$-\frac{1}{3}\lvert 6-5x\rvert + 2 \geq 1$	Add -2 to both sides
$-\frac{1}{3}\lvert 6-5x\rvert \geq -1$	Multiply both sides by -3 and change the inequality
$\lvert 6-5x\rvert \leq 3$	Write equivalent inequality
$-3 \leq 6-5x \leq 3$	Subtract 6 from all three parts
$-9 \leq -5x \leq -3$	Divide all three parts by -5 and change the inequality
$\frac{9}{5} \geq x \geq \frac{3}{5}$	Give interval notation (graph first if it helps)
$\left[\frac{3}{5}, \frac{9}{5}\right]$	Final answer

1.8 Solving Absolute Value Equations and Inequalities Practice

Solve

1. $|2x+3|=5$
2. $|3-x|=7$
3. $|4x-3|=13$
4. $|5-3x|=2$
5. $\left|\frac{3x+4}{3}\right|=5$
6. $\left|\frac{3x+1}{4}\right|=2$
7. $3-|5x-2|=-7$
8. $\frac{5}{2}-\left|\frac{2x+1}{2}\right|=\frac{3}{2}$
9. $\frac{2}{3}-3\left|\frac{x-2}{2}\right|=\frac{1}{6}$
10. $4-2|4-3x|=-8$
11. $|5x+3|=|2x-1|$
12. $|2+3x|=|4-2x|$
13. $|3x-4|=|2x+3|$
14. $\left|\frac{2x-5}{3}\right|=\left|\frac{3x+4}{2}\right|$
15. $\left|\frac{4x-2}{5}\right|=\left|\frac{6x+3}{2}\right|$
16. $\left|\frac{3x+2}{2}\right|=\left|\frac{2x-3}{3}\right|$

Solve. Give answers in interval notation.

17. $|x|<3$
18. $|x|>5$
19. $|x|\le 8$
20. $|2x|<6$
21. $|3x|>5$
22. $|x+3|<4$
23. $|x-2|<6$
24. $|x-4|>5$
25. $|x-8|<12$
26. $|x+3|\le 4$
27. $|x+3|\ge 3$
28. $|x-1|<3$
29. $\left|\frac{5x}{2}\right|\le 5$
30. $\left|\frac{2x}{3}\right|>4$
31. $|3x-2|<9$
32. $|2x-4|>6$
33. $|3x-5|\ge 3$
34. $|2x+5|<9$

35. $\left|x+\frac{2}{3}\right|<\frac{5}{3}$

36. $\left|x-\frac{1}{2}\right|<\frac{3}{2}$

37. $\left|x-\frac{3}{4}\right|<\frac{7}{4}$

38. $\left|\frac{x-3}{2}\right|\geq 3$

39. $\left|\frac{x-2}{2}\right|<1$

40. $\left|\frac{3x-2}{4}\right|>2$

41. $\left|\frac{2x+1}{3}\right|<3$

42. $\left|\frac{2x-5}{3}\right|\geq 3$

43. $1+2|x-1|\leq 9$

44. $3-|2-x|<1$

45. $4+3|x-1|\geq 10$

46. $10-3|x-2|\geq 4$

47. $6-|2x-5|\geq 3$

48. $3-2|3x-1|\geq -7$

49. $\frac{3}{2}-2\left|\frac{x+4}{4}\right|\geq -\frac{3}{2}$

50. $\frac{1}{3}+\left|\frac{2x+1}{6}\right|\geq \frac{1}{2}$

51. $\frac{2}{3}-2\left|\frac{2x-2}{3}\right|\geq -2$

52. $\frac{1}{3}-3\left|\frac{3-x}{2}\right|\geq -\frac{1}{6}$

53. $\frac{2}{3}-\left|\frac{3x-2}{6}\right|\leq -\frac{1}{2}$

54. $\frac{1}{4}-\left|\frac{2x+3}{3}\right|<-\frac{5}{2}$

Chapter 2:

Functions and Graphs

2.1 Functions

One of the two most important concepts in mathematics is the concept of a function. Understanding the concept of a function is central to many of the operations and procedures in advanced mathematics.

Definition of Function

A function is a rule that relates two groups of objects called the Domain and the Range of the function. Under the function rule, an element of the Domain gets paired with one element of the Range.

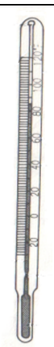
To give an example of a function it is necessary to give three things, the Domain, the Range, and the rule that connects the two. Consider the following example:

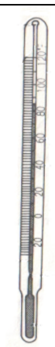
Domain: times of the day

Range: temperature

Rule: look at the thermometer

On a given day we can apply the function with the following results given as a table:

Time AM		Temperature
4:00		35° F
5:00		40 ° F
6:00		42 ° F
7:00		45 ° F
8:00		53 ° F
9:00		55 ° F
10:00		60 ° F
11:00		70 ° F

Time PM		Temperature
12:00		70 ° F
1:00		72 ° F
2:00		68 ° F
3:00		65 ° F
4:00		62 ° F
5:00		62 ° F
6:00		60 ° F
7:00		55 ° F

Notice that for each time (Domain value) there is only one temperature (Range value) as required by the definition. But also notice that the converse does not hold. The same temperature can occur at multiple times. For example, the temperature 55 ° occurs at both 9:00 AM and at 7:00 PM. Our definition of a function does not prohibit this from happening. That is, while each element of the Domain can be paired with only one element of the Range, each element of the range can be paired with multiple elements of the Domain.

Definition of One-to-One:

A function where each element of the Range is paired with exactly one element of the Domain is called one-to-one.

The standard notation for a function is $y = f(x)$ where the letters x and y stand for the Domain and Range variables respectively. The letter f stands for the function rule. As in algebra the letters x, y, and z usually represent variables, in function theory they also represent variables. In function theory the letters f, g, and h usually represent the function rule. Although in applications other letters may be used for mnemonic purposes. For example, in economics a demand function may be represented by $q = d(p)$ where q represents a quantity of a commodity, p represents the price, and the function d represents the demand or relationship between the price and the quantity available in the market place.

The variable, x, that represents the Domain value is frequently called the independent variable. It is called independent because there is a sense of choice involved in picking it. In the above example relating time and temperature, the Domain, time, is independent because we can choose the time to look at the thermometer. The Range values are called dependent because whatever y value we get from the function is dependent upon two things, the domain value we choose and the function we are using.

In algebra functional relationships are regularly given by algebraic expressions. An example is $y = f(x) = 3x^2 - 2x + 4$. The '$y =$' at the front is awkward and is frequently left out and we write $f(x) = 3x^2 - 2x + 4$. But, every function has a range value. It must be remembered that the '$y =$' is there even if it isn't written. In some advanced texts you might see $y(x) = 3x^2 - 2x + 4$ where the letter y represents both the range value and the function name. While a useful contraction it should be remembered that the function value, y, and the function name, f, are different things.

In this text we will be primarily interested in functions that are represented by algebraic relationships.

Evaluation of a function.

Evaluating a function is simply a process of substitution. While we might define a function as $f(x) = 3x^2 + 5x - 2$, the x in the definition is simply a hole that we fill with the domain letter x. This function could also be written as $f(\square) = 3\square^2 + 5\square - 2$ where the $\square$ is a hole that gets filled with something (the same thing in each $\square$).

To evaluate the function $f(x) = 3x^2 + 5x - 2$ with $x = 2$ or $f(2)$ (read f at 2 or f of 2) we have $f(2) = 3(2)^2 + 5(2) - 2$. Each of the x's in f have been replaced by 2.

Example 1: Evaluate $f(2)$ given $f(x) = 3x^2 + 5x - 2$

$f(x) = 3x^2 + 5x - 2$	Substitute a 2 for each x
$f(2) = 3(2)^2 + 5(2) - 2$	Simplify exponents
$f(2) = 3(4) + 5(2) - 2$	Multiply
$f(2) = 12 + 10 - 2$	Add
$f(2) = 20$	Final answer

As shown in example 1, evaluating this expression gives $f(2) = 20$ or $y = 20$ when $x = 2$ (remember, there is always a y or range value present even if we don't write it as part of the function).

The advantage of function notation is that it allows us to write more complicated substitutions.

Example 2: Evaluate $f(a + 5)$ if $f(x) = 3x^2 + 5x - 2$

$f(x) = 3x^2 + 5x - 2$	Substitute $(a + 5)$ for each x
$f(a + 5) = 3(a + 5)^2 + 5(a + 5) - 2$	Simplify exponent
$f(a + 5) = 3(a^2 + 10a + 25) + 5(a + 5) - 2$	Distribute
$f(a + 5) = 3a^2 + 30a + 75 + 5a + 25 - 2$	Combine like terms
$f(a + 5) = 3a^2 + 35a + 98$	Final answer

Example 3: Evaluate $f(x - h)$ if $f(x) = 3x^2 + 5x - 2$

$f(x) = 3x^2 + 5x - 2$	Substitute $(x - h)$ for each x
$f(x - h) = 3(x - h)^2 + 5(x - h) - 2$	Simplify exponent
$f(x - h) = 3(x^2 - 2hx + h^2) + 5(x - h) - 2$	Distribute
$f(x - h) = 3x^2 - 6hx + 3h^2 + 5x - 5h - 2$	Final answer

Example 4: Evaluate $h(x+1)$ if $h(x) = x + \frac{1}{x}$

$$h(x) = x + \frac{1}{x}$$ Substitute $(x+1)$ for each x

$$h(x+1) = x + 1 + \frac{1}{x+1}$$ Final answer

Example 5: Evaluate $h\left(\frac{1}{x}\right)$ if $h(x) = x + \frac{1}{x}$

$$h(x) = x + \frac{1}{x}$$ Substitute $\frac{1}{x}$ for each x

$$h\left(\frac{1}{x}\right) = \frac{1}{x} + \frac{1}{\frac{1}{x}}$$ For compound fraction, multiply 2^{nd} term by reciprocal of $\frac{1}{x}$

$$h\left(\frac{1}{x}\right) = \frac{1}{x} + x$$ Final answer

Domain of a Function

The Domain of a function is the collection of all elements that make sense when the function is evaluated with them.

Example 6: Let $f(x) = \frac{x+1}{x-1}$. Is $x = -1$ in the Domain of the function?

$$f(x) = \frac{x+1}{x-1}$$ Substitute -1 for each x

$$f(-1) = \frac{(-1)+1}{(-1)-1}$$ Simplify

$$f(-1) = \frac{0}{-2} = 0$$ Because this makes sense –1 is in the Domain of the function.

Example 7: Let $f(x)=\frac{x+1}{x-1}$. Is $x=1$ in the Domain of the function?

$$f(x)=\frac{x+1}{x-1}$$ Substitute 1 for each x

$$f(1)=\frac{(1)+1}{(1)-1}$$ Simplify

$$f(1)=\frac{2}{0}$$ Because division by zero is undefined this does not make sense and x = 1 is not in the Domain.

Example 8: Let $g(x)=\sqrt[4]{5-x}$. What is the domain?

Because this is an even root and even roots have difficulties with negative values the expression inside the radical has to be ≥ 0. Or, $5-x\geq 0$.

$$5-x\geq 0$$ Subtract 5

$$-x\geq -5$$ Divide by -1, flip inequality

$$x\leq 5$$ Final answer

Example 9: Let $h(x)=\frac{\sqrt{2x+4}}{x^2-x-2}$. What is the domain?

This function has both an even root and a denominator. We know that even roots have difficulties with negative values and the denominator cannot equal zero.

$$2x+4\geq 0$$ First we consider the even root ≥ 0, Subtract 4

$$2x\geq -4$$ Divide by 2

$$x\geq -2$$ Now consider the denominator

$$x^2-x-2\neq 0$$ Factor

$$(x-2)(x+1)\neq 0$$ Set each factor equal to zero

$$x-2\neq 0 \quad \text{or} \quad x+1\neq 0$$ Solve each equation

$$x\neq 2,-1$$ Put both parts together for domain

$$x\geq -2 \text{ and } x\neq 2,-1$$ Final answer

Range of a Function

A value is inside the Range of the function if there is a value in the Domain that gets paired with it. Remember, a function rule pairs a Domain value with a Range value. That is, for every value in the Domain there is a corresponding value in the Range. So, a value can get in the Range only if there is something in the Domain to pair it with.

Example 10: If $f(x) = 4 - x^2$, is the value $y = 0$ in the Range of f?

$f(x) = 4 - x^2$	y is equivalent to $f(x)$, substitute 0 for $f(x)$
$0 = 4 - x^2$	Factor
$0 = (2 - x)(2 + x)$	Set each factor equal to zero
$2 - x = 0$ or $2 + x = 0$	Solve each equation
$x = \pm 2$	We have a solution, so yes, $y = 0$ is in the Range.

Example 11: If $f(x) = 4 - x^2$, is the value $y = 5$ in the Range of f?

$f(x) = 4 - x^2$	Substitute 5 for $f(x)$
$5 = 4 - x^2$	Subtract 4
$1 = -x^2$	Divide by -1
$-1 = x^2$	Square root both sides
$\pm\sqrt{-1} = x$	This equation has only imaginary roots

We must conclude that there are no Domain values that get paired with 5 and that 5 is not in the Range of the function.

Example 12: If $h(x) = \frac{x-1}{x+1}$, is the value $y = -1$ in the Range of the function?

$h(x) = \dfrac{x-1}{x+1}$	Substitute -1 for $h(x)$
$-1 = \dfrac{x-1}{x+1}$	Multiply by $(x+1)$ to clear the denominator
$-1(x+1) = x-1$	Distribute
$-x-1 = x-1$	Add x and add 1 to both sides
$0 = 2x$	Divide by 2
$0 = x$	Because we found a value of x that will get paired with -1, it must be in the range of h(x)

Example 13: If $h(x) = \frac{x-1}{x+1}$, is the value $y = 1$ in the range of the function?

$h(x) = \dfrac{x-1}{x+1}$	Substitute 1 for $h(x)$
$1 = \dfrac{x-1}{x+1}$	Multiply by $(x+1)$ to clear the denominator
$1(x+1) = (x-1)$	Distribute
$x+1 = x-1$	Subtract x
$1 \neq -1$	The variable has dropped out of the equation and the remaining statement is false

The equation has no solutions. This means that there does not exist a value of x that will get paired with 1, therefore 1 cannot be in the range of $h(x)$.

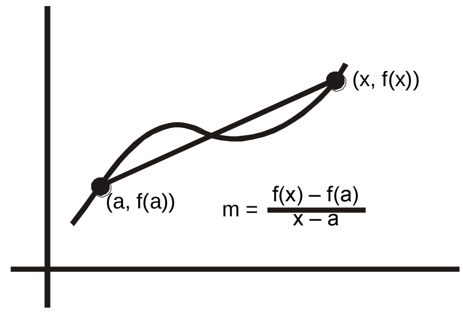
Notice with this example that $x = -1$ is not in the Domain of $h(x)$. This would put a zero in the denominator. But $y = -1$ is in the Range of $h(x)$. The fact that a value is or is not in the Domain of a function has nothing to do with whether the value is or is not in the Range of the function.

Difference Quotient

A common expression used in mathematics is the Difference Quotient. There are three versions of the Difference quotient. Given a function $f(x)$ they are:

$$\frac{f(x)-f(a)}{x-a} \qquad \frac{f(x+h)-f(x)}{h} \qquad \frac{f(x+h)-f(x-h)}{2h}$$

The Difference Quotient represents the average rate of change between two points on a function, or the slope of the secant line connecting two points. This is illustrated in the graph to the right.



Example 14: Let $f(x) = 2x^2 - 3x$. Evaluate the difference Quotient $\frac{f(x+h)-f(x)}{h}$

$\frac{f(x+h)-f(x)}{h}$	We need to find $f(x+h)$
$f(x+h) = 2(x+h)^2 - 3(x+h)$	Simplify the exponent
$f(x+h) = 2(x^2+2hx+h^2) - 3(x+h)$	Distribute
$f(x+h) = 2x^2+4hx+2h^2-3x-3h$	Substitute this and $f(x)$ into the Difference Quotient
$\frac{(2x^2+4hx+2h^2-3x-3h)-(2x^2-3x)}{h}$	Distribute the negative
$\frac{2x^2+4hx+2h^2-3x-3h-2x^2+3x}{h}$	Combine like terms
$\frac{4hx+2h^2-3h}{h}$	Factor h out of numerator
$\frac{h(4x+2h-3)}{h}$	Divide out h
$4x+2h-3$	Final answer

Example 15: Let $g(x) = x^3 + x$. Evaluate the Difference Quotient $\frac{g(x+h)-g(x-h)}{2h}$.

$\dfrac{g(x+h)-g(x-h)}{2h}$	We need to find $g(x+h)$ and $g(x-h)$
$g(x+h) = (x+h)^3 + (x+h)$	First, $g(x+h)$. Simplify exponent
$g(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 + x + h$	Next, $g(x-h)$
$g(x-h) = (x-h)^3 + (x-h)$	Simplify exponent
$g(x-h) = x^3 - 3x^2h + 3xh^2 - h^3 + x - h$	Substitute both into Difference Quotient
$\dfrac{(x^3 + 3x^2h + 3xh^2 + h^3 + x + h) - (x^3 - 3x^2h + 3xh^2 - h^3 + x - h)}{2h}$	Distribute negative
$\dfrac{x^3 + 3x^2h + 3xh^2 + h^3 + x + h - x^3 + 3x^2h - 3xh^2 + h^3 - x + h}{2h}$	Combine like terms
$\dfrac{6x^2h + 2h^3 + 2h}{2h}$	Factor $2h$ in numerator
$\dfrac{2h(3x^2 + h^2 + 1)}{2h}$	Divide out $2h$
$3x^2 + h^2 + 1$	Final answer

2.1 Functions Practice

1. Specify the domain of each of the following functions

a. $f(x) = -5x + 1$

b. $f(x) = |x| + 3$

c. $f(x) = x^2 - 3x - 4$

d. $f(x) = \sqrt{x}$

e. $f(x) = x^4$

f. $f(x) = \sqrt{x - 16}$

g. $f(x) = \dfrac{x}{x - 3}$

h. $f(x) = -\dfrac{2}{x^2 - 3x - 4}$

i. $f(x) = 4$

j. $y(x) = 4x - 5$

k. $y(x) = \dfrac{x - 4}{x + 4}$

l. $h(x) = \dfrac{\sqrt{3x - 12}}{x^2 - 25}$

m. $s(t) = \dfrac{1}{t^2}$

n. $y(x) = \dfrac{x}{x^2 - 25}$

o. $p(s) = \dfrac{1}{s^2 - 4s}$

p. $s(t) = \dfrac{1}{t^2 + 1}$

q. $f(x) = \sqrt{5 - 4x}$

r. $z(t) = \dfrac{t}{|t|}$

2. Let $f(x) = 4 - x^2$
   a. Is $y = 2$ in the Range of f?
   b. Is $y = 13$ in the Range of f?

3. Let $f(x) = \sqrt{9 + x^2}$
   a. Is $y = 9$ in the Range of f?
   b. Is $y = 1$ in the Range of f?

4. Let $f(x) = \frac{3x}{2+x^2}$
   a. Is $y = 1$ in the Range of f?
   b. Is $y = 2$ in the Range of f?

5. Let $f(x) = \frac{2x-1}{x+4}$
   a. What is the Domain of f?
   b. Is $y = 3$ in the Range of f?
   c. Is $y = 2$ in the Range of f?

6. Let $f(x) = \frac{x-5}{x+5}$
   a. What is the Domain of f?
   b. Is $y = -2$ in the Range of f?
   c. Is $y = 1$ in the Range of f?

7. Let $f(x) = \frac{2x+1}{4x-1}$
   a. What is the Domain of f?
   b. Is $y = \frac{1}{2}$ in the Range of f?
   c. Is $y = 2$ in the Range of f?

8. Let $f(x) = \frac{ax-b^2}{x-a}$
   a. What is the Domain of f?
   b. Is $y = b$ in the Range of f?
   c. Is $y = a$ in the Range of f?

9. Let $f(x) = x^2 - 3x + 1$. Evaluate each:

a.	$f(1)$	b.	$f(0)$	c.	$f(-1)$
d.	$f\left(\frac{3}{2}\right)$	e.	$f(z)$	f.	$f(x+1)$
g.	$f(a+1)$	h.	$f(-x)$	i.	$\lvert f(1)\rvert$
j.	$f(\sqrt{3})$	k.	$f(1+\sqrt{2})$	l.	$\lvert 1 - f(2)\rvert$

10. Let $H(x) = 1 - x - x^2 - x^3$

a. Which is larger, $H(0)$ or $H(1)$?

b. Find $H\left(\frac{1}{2}\right)$. Does $H\left(\frac{1}{2}\right) + H\left(\frac{1}{2}\right) = H(1)$?

11. Let $f(x) = 3x^2$. Evaluate each:

a.	$f(2x)$	b.	$2f(x)$	c.	$f(x^2)$
d.	$[f(x)]^2$	e.	$f\left(\frac{x}{2}\right)$	f.	$\frac{f(x)}{2}$

12. Let $f(x) = 4 - 3x$. Evaluate each:

a.	$f(2)$	b.	$f(3)$	c.	$f(2) + f(3)$
d.	$f(2+3)$	e.	$f(2x)$	f.	$2f(x)$
g.	$f(x^2)$	h.	$f\left(\frac{1}{x}\right)$	i.	$f[f(x)]$
j.	$x^2 f(x)$	k.	$\frac{1}{f(x)}$	l.	$f(-x)$
m.	$-f(x)$	n.	$-f(-x)$		

13. Let $H(x) = 1 - 2x^2$. Evaluate each:

a.	$H(0)$	b.	$H(2)$	c.	$H(\sqrt{2})$

d. $H\left(\frac{5}{6}\right)$

e. $H(1-\sqrt{3})$

f. $H(x^2)$

g. $H(x+1)$

h. $H(x+h)$

i. $H(x+h)-H(x)$

14. Let $R(x)=\frac{2x-1}{x-2}$. Evaluate each:

a. $R(2)$

b. $R(0)$

c. $R\left(\frac{1}{2}\right)$

d. $R(-1)$

e. $R(x^2)$

f. $R\left(\frac{1}{x}\right)$

g. $R(a)$

h. $R(x-1)$

15. Let $g(x)=2$ for all x. Evaluate each:

a. $g(0)$

b. $g(5)$

c. $g(x+h)$

16. Let $d(t)=-16t^2+96t$

a. Compute $d(1), d\left(\frac{3}{2}\right), d(2)$, and $d(t_0)$

b. For which values of t is $d(t)=0$?

c. For which values of t is $d(t)=1$?

17. Let $A(x)=|x^2-1|$. Compute $A(2), A(1)$, and $A(0)$.

18. Let $g(t)=|t-4|$. Find $g(3)$. Find $g(x+4)$.

19. Let $f(x)=\frac{x^2}{|x|}$.

a. What is the domain of f?

b. Find $f(2), f(-2)$, and $f(-20)$.

20. Let $G(x)=3x-5$. Compute each:

a. $\dfrac{G(x)-G(a)}{x-a}$

b. $\dfrac{G(x)-G(x_0)}{x-x_0}$

21. Let $H(x)=1-\frac{x}{4}$. Compute $\frac{H(x)-H(1)}{x-1}$

22. Let $f(x)=x^2$. Compute $\frac{f(b)-f(a)}{b-a}$ using the values $a=5$ and $b=3$.

23. Let $g(x) = x^2 - 2x + 1$. Compute the following:

a. $\dfrac{g(1+h) - g(2)}{h}$ b. $\dfrac{g(x) - g(1)}{x - 1}$ c. $\dfrac{g(1+h) - g(1-h)}{2h}$

24. Let $f(t) = t^2 + t$. Compute the following:

a. $\dfrac{f(2+h) - f(2)}{h}$ b. $\dfrac{f(t) - f(2)}{t - 2}$ c. $\dfrac{f(2+t) - f(2-t)}{4}$

25. Let $M(x) = \frac{x-a}{x+a}$. Find $M\left(\frac{1}{x}\right)$

26. Let $g(t) = t^4 - 3t^2 + 1$. Find $g(\sqrt{t})$

27. Let $k(x) = 5x^3 + \frac{5}{x^3} - x - \frac{1}{x}$. Find $k\left(\frac{1}{x}\right)$

28. Let $f(x) = \frac{5x-7}{2x+1}$. Find $\frac{f(x+h)-f(x)}{h}$.

29. Let $(t) = \frac{t-x}{t+x}$. Find $f(x+y) + f(x-y)$.

30. Let $(u) = \frac{2u-1}{u+3}$. Find $H\left(\frac{u}{4}\right)$.

31. Let $f(z) = \frac{3z-4}{5z-3}$. Find $f\left(\frac{3z-4}{5z-3}\right)$.

32. Let $F(x) = \frac{ax+b}{cx-a}$. Find $F\left(\frac{ax+b}{cx-a}\right)$.

33. If $f(x) = -2x^2 + 6x + k$ and $f(0) = -1$, find k.

34. If $g(x) = x^2 - 3kx - 4$ and $g(1) = -2$, find k

35. Let $f(x) = x^2 - 5x - 6$

a. Find all values of x for which $f(x) = 0$

b. Find all values of x for which $f(x) = 1$

c. Find all values of x for which $f(x) = -15$

36. Find $\frac{f(x+h)-f(x)}{h}$ for each of the following functions

a. $f(x) = x^2$

b. $f(x) = x^2 + 1$

c. $f(x) = x^2 + c$

d. $f(x) = x^3$

e. $f(x) = x^3 + 1$

f. $f(x) = x^3 + c$

g. $f(x) = \frac{1}{x}$

h. $f(x) = \sqrt{x}$

i. $f(x) = \frac{1}{\sqrt{x}}$

j. $f(x) = -3x + 6$

k. $f(x) = 7$

l. $f(x) = 2x^2 - 3x + 1$

37. Find $\frac{g(x+h)-g(x-h)}{2h}$ for each of the following functions

a. $g(x) = x + 1$

b. $g(x) = x^2 - 1$

c. $g(x) = x^2 - x$

d. $g(x) = \frac{1}{x}$

e. $g(x) = \sqrt{x}$

f. $g(x) = \frac{1}{\sqrt{x}}$

2.2 Algebra of Functions

Operations on Functions

Consider the two algebraic expressions $y = x^2 - x - 2$ and $z = x^2 - 4$.

Find each of the following combinations of y and z:

a. $y + z = (x^2 - x - 2) + (x^2 - 4) = x^2 - x - 2 + x^2 - 4 = 2x^2 - x - 6$
b. $y - z = (x^2 - x - 2) + (x^2 - 4) = x^2 - x - 2 - x^2 + 4 = -x + 2$
c. $yz = (x^2 - x - 2)(x^2 - 4) = x^4 - 4x^2 - x^3 + 4x - 2x^2 + 8 = x^4 - x^3 - 6x^2 + 4x + 8$
d. $\frac{y}{z} = \frac{x^2-x-2}{x^2-4} = \frac{(x+1)(x-2)}{(x+2)(x-2)} = \frac{x+1}{x+2}$

Now consider the two functions, $f(x) = x^2 - x - 2$ and $g(x) = x^2 - 4$. Just as we can add, subtract, multiply and divide the algebraic expressions we can add, subtract, multiply and divide the functions. Therefore,

a. $f(x) + g(x) = (x^2 - x - 2) + (x^2 - 4) = x^2 - x - 2 + x^2 - 4 = 2x^2 - x - 6$

b. $f(x) - g(x) = (x^2 - x - 2) - (x^2 - 4) = x^2 - x - 2 - x^2 + 4 = -x + 2$

c. $f(x)g(x) = (x^2 - x - 2)(x^2 - 4) = x^4 - 4x^2 - x^3 + 4x - 2x^2 + 8 = x^4 - x^3 - 6x^2 + 4x + 8$

d. $\frac{f(x)}{g(x)} = \frac{x^2-x-2}{x^2-4} = \frac{(x+1)(x-2)}{(x+2)(x-2)} = \frac{x+1}{x+2}$

In effect, the basic arithmetic operations on functions don't change anything. But there are some subtle differences. Because arithmetic expressions deal with static value of the variable and functions deal with domains, when dividing our functions we need to be careful that we don't divide by zero. That is, if zero is in the range of the denominator we must exclude that value from the quotient. In the above example, $\frac{f(x)}{g(x)} = \frac{x+1}{x+2}$ provided that $x \neq \pm 2$. For either of these two values the denominator in the quotient is zero and the quotient is undefined.

There is also a small change in the notation. For $f(x) + g(x)$ we write the sum function $(f + g)(x)$ and for the difference $f(x) - g(x)$ we write the difference function $(f - g)(x)$. This is not an application of the distributive property. We are not factoring out the x even though it looks that way. We are simply changing the notation putting the emphasis of the operation on the f and g. In a similar vein for the product and quotient we write $(fg)(x)$ for $f(x)g(x)$ and $\left(\frac{f}{g}\right)(x)$ for $\frac{f(x)}{g(x)}$. Thus, the sum, difference, product and quotient of two functions is given by $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$ and these operations are performed by adding, subtracting, multiplying and dividing the two functions.

Composition of Two Functions

Another operation that we perform with functions is called *composition* of the functions. To introduce this notion, let us recall first the techniques of using functional notation.

If $f(x) = x^2 - 1$ and $g(x) = x + 6$, then $f(3)$ means "replace x by 3" in $f(x)$.

So $f(3) = 3^2 - 1 = 9 - 1 = 8$. $f(a)$ means "replace x by a" in $f(x)$, so $f(a) = a^2 - 1$. $f(a+b)$ means "replace x by $a + b$ " in $f(x)$, so $f(a+b) = (a+b)^2 - 1$.

Now, looking at the above illustrations, what is the only possible meaning of $f(g(x))$? Think about it. It can only mean "replace x by $g(x)$" in $f(x)$. But: $g(x) = x + 6$.

Example: Let $f(x) = x^2 - 1$ and $g(x) = x + 6$. Find $f\big(g(x)\big)$

$f\big(g(x)\big)$	Replace x by $g(x)$, or $(x+6)$ in $f(x)$
$f\big(g(x)\big) = (x+6)^2 - 1$	Square the binomial
$f\big(g(x)\big) = x^2 + 12x + 36 - 1$	Combine like terms
$f\big(g(x)\big) = x^2 + 12x + 35$	Final answer

The symbol, $f(g(x))$, *means the composition of* f *with* g.

In like manner, the composition of g with f would mean $g(f(x))$, which tells us to "replace x by $f(x)$" in $g(x)$.

Example 2: Let $f(x) = x^2 - 1$ and $g(x) = x + 6$. Find $g(f(x))$

$g\big(f(x)\big)$	Replace x by $f(x)$, or $x^2 - 1$ in $g(x)$
$(x^2 - 1) + 6$	Combine like terms
$x^2 + 5$	Final answer

CAUTION!

Note that in the foregoing examples $f(g(x))$ and $g(f(x))$ are two different functions; i.e., $f(g(x)) \neq g(f(x))$, and this is usually the case.

Example 3: Let $f(x) = \sqrt{x+1}$ and $g(x) = x^2 - 2x$. Find $f\big(g(x)\big)$

$f\big(g(x)\big)$ Replace x by $g(x)$, or $x^2 - 2x$ in $f(x)$

$f\big(g(x)\big) = \sqrt{(x^2 - 2x) + 1}$ Clear parentheses

$f\big(g(x)\big) = \sqrt{x^2 - 2x + 1}$ Factor perfect square trinomial

$f\big(g(x)\big) = \sqrt{(x-1)^2}$ Clear the inverse radical and exponent

$f\big(g(x)\big) = x - 1$ Final answer

Example 4: Let $f(x) = \sqrt{x+1}$ and $g(x) = x^2 - 2x$. Find $g\big(f(x)\big)$

$g\big(f(x)\big)$ Replace x by $f(x)$, or $\sqrt{x+1}$ in $g(x)$

$g\big(f(x)\big) = \left(\sqrt{x+1}\right)^2 - 2\left(\sqrt{x+1}\right)$ Clear inverse radical and exponent

$g\big(f(x)\big) = x + 1 - 2\sqrt{x+1}$ Final answer

Example 5: Let $f(x) = x^2 + x$ and $g(x) = \frac{1}{x+2}$. Find $f\big(g(x)\big)$

$f\big(g(x)\big)$ Replace x by $g(x)$, or $\frac{1}{x+2}$ in $f(x)$

$f\big(g(x)\big) = \left(\frac{1}{x+2}\right)^2 + \frac{1}{x+2}$ Multiply second fraction by $(x+2)$ to get common denominator

$f\big(g(x)\big) = \frac{1}{(x+2)^2} + \frac{x+2}{(x+2)^2}$ Add numerators

$f\big(g(x)\big) = \frac{x+3}{(x+2)^2}$ Final answer

Example 6: Let $f(x) = x^2 + x$ and $g(x) = \frac{1}{x+2}$. Find $g\big(f(x)\big)$

$g\big(f(x)\big)$ Replace x by $f(x)$, or $x^2 + x$ in $g(x)$

$$g\big(f(x)\big) = \frac{1}{(x^2 + x) + 2}$$ Clear parentheses

$$g\big(f(x)\big) = \frac{1}{x^2 + x + 2}$$ Final answer

Decomposition

After one learns to multiply two expressions together they are taught to factor an expression into two factors. Similarly, just as we can factor an algebraic expression we can decompose the composition of two functions. Consider the function $h(x) = (x + 2)^2$. We are required to find two functions $f(x)$ and $g(x)$ such that $f(g(x)) = h(x)$. The basic process is to look at the pattern and determine if there is an obvious outer function or an obvious inner function. By inner or outer function we are referring to the fact that in the expression $f(g(x))$ the function g is inside the function f. For $h(x) = (x + 2)^2$ the obvious outer function is squaring or x^2. That is, we have an expression, $x + 2$, that is being squared. Similarly, the obvious inner function is $x + 2$. Consequently, we can let the outer function $f(x) = x^2$ and the inner function $g(x) = x + 2$ Or, $h(x) = f\big(g(x)\big) = (x + 2)^2$.

Example 7: Decompose $h(x) = \sqrt{\frac{x+1}{x-1}}$ into two function $f(x)$ and $g(x)$ so $f\big(g(x)\big) = h(x)$

$$h(x) = \sqrt{\frac{x+1}{x-1}}$$ Note outer function is the square root

$f(x) = \sqrt{x}$ Everything inside the function, is inner function

$$g(x) = \frac{x+1}{x-1}$$ These are our functions such that $f\big(g(x)\big) = h(x)$

Example 8: Decompose $h(x) = (x^2 - 1)^{1/3} - (x^2 - 1)^{2/3}$ into two functions $f(x)$ and $g(x)$ so that $f\big(g(x)\big) = h(x)$

$h(x) = (x^2 - 1)^{2/3} - (x^2 - 1)^{1/3}$ Inside function stands out, $x^2 - 1$, is repeated

$g(x) = x^2 - 1$ Outside function is what happens to this

$f(x) = x^{2/3} - x^{1/3}$ These are our functions such that $f\big(g(x)\big) = h(x)$

Example 9: Decompose $p(x) = \left(\frac{x^2+1}{x^2-1}\right)^{3/4}$ into three functions, $f(x)$, $g(x)$, and $h(x)$ such that $f\left(g\big(h(x)\big)\right) = p(x)$

$p(x) = \left(\dfrac{x^2+1}{x^2-1}\right)^{\frac{3}{4}}$	The outer function is clear with the $\frac{3}{4}$ power
$f(x) = x^{3/4}$	The inner function is repeated, x^2
$h(x) = x^2$	The middle function is what happens to x^2
$g(x) = \dfrac{x+1}{x-1}$	These are our functions such that $f\left(g\big(h(x)\big)\right) = p(x)$

2.2 Algebra of Functions Practice

1. Let $f(x) = 2x - 3$ and $g(x) = x^2 + 1$. Find:

a. $(f + g)(5)$ b. $(f - g)(3)$ c. $(fg)(2)$

d. $\left(\frac{f}{g}\right)(4)$ e. $(f + g)(x)$ f. $(f - g)(x)$

g. $(fg)(x)$ h. $\left(\frac{f}{g}\right)(x)$

2. Let $f(x) = \frac{x-2}{x+1}$ and $g(x) = x^2 - x - 2$. Find:

a. $(f + g)(2)$ b. $(f - g)(5)$ c. $(fg)(102)$

d. $\left(\frac{f}{g}\right)(99)$ e. $(f + g)(x)$ f. $(f - g)(x)$

g. $(fg)(x)$ h. $\left(\frac{f}{g}\right)(x)$

3. Let $f(x) = \frac{2x^2-x-3}{x-2}$ and $g(x) = x^2 - x - 2$. Find:

a. $(f + g)(1)$ b. $(f - g)(3)$ c. $(fg)(2)$

d. $\left(\frac{f}{g}\right)(0)$ e. $(f + g)(x)$ f. $(f - g)(x)$

g. $(fg)(x)$ h. $\left(\frac{f}{g}\right)(x)$

4. For each of the following pairs of functions, find $f\big(g(x)\big)$ and $g\big(f(x)\big)$.

a. $f(x) = 2x - 3;\ \ g(x) = \dfrac{x+3}{2}$ b. $f(x) = x^2;\ \ g(x) = x - 1$

c. $f(x) = 3x + 2;\ \ g(x) = x^2 - 8$ d. $f(x) = \sqrt{x^2 + 1};\ \ g(x) = \dfrac{x^2}{x^2 - 1}$

e. $f(x) = x^2 + 1;\ \ g(x) = \sqrt{x^2 - 4x}$ f. $f(x) = \dfrac{x-1}{x};\ \ g(x) = x + \dfrac{1}{x}$

g. $f(x) = x + 1;\ \ g(x) = x + 4$ h. $f(x) = 8x;\ \ g(x) = 8 + x$

i. $f(x) = x^3;\ \ g(x) = x^2 + x$

j. $f(x) = 2x + 3;\ \ g(x) = \dfrac{x-3}{2}$

k. $f(x) = x^2 + 3x;\ \ g(x) = x - 4$

l. $f(x) = \sqrt{x};\ \ g(x) = x - 1$

m. $f(x) = \dfrac{1}{x};\ \ g(x) = 3$

n. $f(x) = \dfrac{1}{x};\ \ g(x) = x^2 - 1$

o. $f(x) = x^2 + 1;\ \ g(x) = x^2 - 1$

p. $f(x) = \sqrt{x+1};\ \ g(x) = x - 1$

q. $f(x) = x^2 - x;\ \ g(x) = x + 1$

r. $f(x) = x^2 - 1;\ \ g(x) = x^2 + 1$

s. $f(x) = \sqrt{x^2+1};\ \ g(x) = \sqrt{x^2-1}$

t. $f(x) = \sqrt[3]{x^2+1};\ \ g(x) = x^2 - 1$

u. $f(x) = \sqrt[3]{x-1};\ \ g(x) = x^3 + 1$

v. $f(x) = \dfrac{x}{x-1};\ \ g(x) = \dfrac{x+1}{x}$

w. $f(x) = \dfrac{x+1}{x-1};\ \ g(x) = \dfrac{x-1}{x+1}$

x. $f(x) = \dfrac{2x+1}{x-3};\ \ g(x) = \dfrac{3x+1}{x+2}$

y. $f(x) = \dfrac{3x+1}{x-2};\ \ g(x) = \dfrac{x+1}{x-1}$

z. $f(x) = \dfrac{2x+3}{3x-1};\ \ g(x) = \dfrac{x+3}{3x-2}$

5. Let $f(x) = x^2$, $g(x) = \sqrt{x}$, and $h(x) = x + 1$. Find:

a. $f\Big(g\big(h(x)\big)\Big)$

b. $f\Big(h\big(g(x)\big)\Big)$

c. $g\Big(h\big(f(x)\big)\Big)$

6. For each of the following functions, $h(x)$, find two non-trivial functions $f(x)$ and $g(x)$ so that $h(x) = f\big(g(x)\big)$. Some problems may have more than one answer.

a. $h(x) = (2x+1)^2$

b. $h(x) = (1-x)^3$

c. $h(x) = \sqrt[3]{x^2-4}$

d. $h(x) = \sqrt{9-x}$

e. $h(x) = \dfrac{1}{x-2}$

f. $h(x) = \dfrac{4}{(5x+2)^2}$

g. $h(x) = (x+3)^{\frac{3}{2}}$

h. $h(x) = \dfrac{x^3-1}{x^3+1}$

i. $h(x) = |9x^2 + 6x + 1|$

j. $h(x) = \left(\dfrac{2+x^3}{2-x^3}\right)^6$

k. $h(x) = \sqrt{\dfrac{x-5}{x+2}}$

l. $h(x) = \sqrt{1+\sqrt{1+x}}$

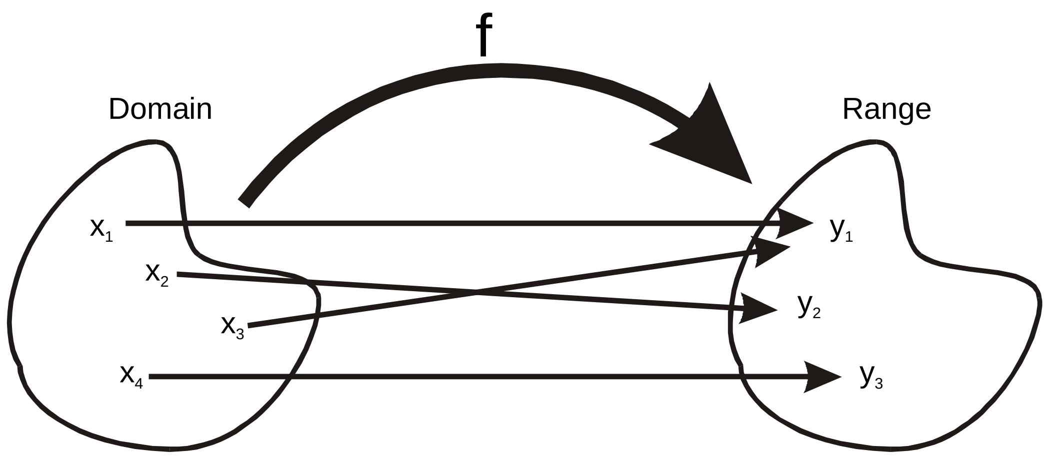
m. $h(x) = (x+4)^2 + 2(x+4)$

n. $h(x) = 4(x-1)^{2/3} + 5 - (x+3)^2 + 4(x+3)$

2.3 Inverse Functions

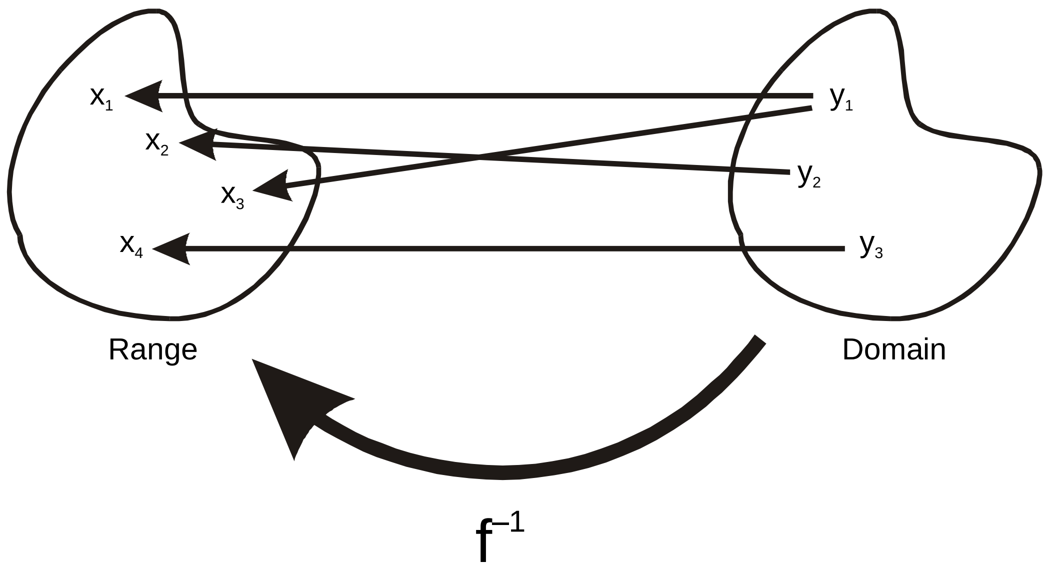
The inverse of an operation in mathematics is to undo something that you did. For example, you can tie a knot in your shoelaces, which, while inconvenient, can nonetheless be untied. The inverse of tying a knot is untying it. Most mathematical operations have the property of being inverted. Suppose that we start with some number, x. We can change the value of the expression by adding a number to it, say $x + 2$. What do we do in order to get back to the x? We take the expression $x + 2$ and subtract 2, that is, $x + 2 - 2 = x$. This works because subtraction is the inverse operation of addition. Similarly, if we multiply x by 2 to get $2x$, to get back to x we divide by 2, or $\frac{2x}{2} = x$. Again, division is the inverse operation of multiplication. Also, if we raise x to a power, x^n, we need only take the n^{th} root. That is, $\sqrt[n]{x^n} = x$.

We want to extend this idea of inverting an operation to the idea of a function. Consider the function given in the diagram below.



Here the function f take the elements of its domain $\{x_1, x_2, x_3, x_4\}$ and pairs them with the range elements $\{y_1, y_2, y_3\}$ giving us the ordered pairs $(x_1, y_1), (x_2, y_2), (x_3, y_1), (x_4, y_3)$.

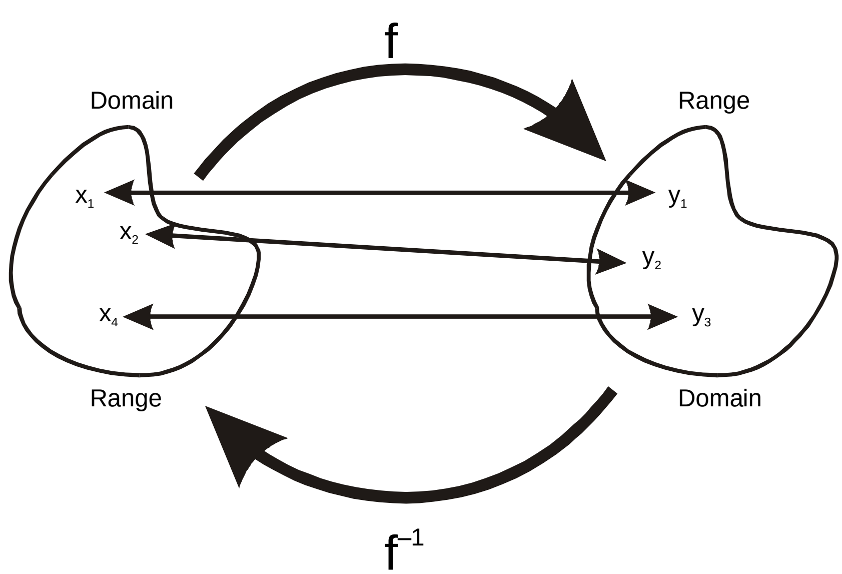
The inverse of f, written f^{-1}, will reverse the direction, that is will take y_1 to x_1, y_2 to x_2, y_1 to x_3, y_3 to x_4. This gives us the ordered pairs $(y_1, x_1), (y_2, x_2), (y_1, x_3), (y_3, x_4)$. This pairing is illustrated in the following diagram. Note that the Range of f is the Domain of f^{-1} and the Domain of f is the Range of f^{-1}.



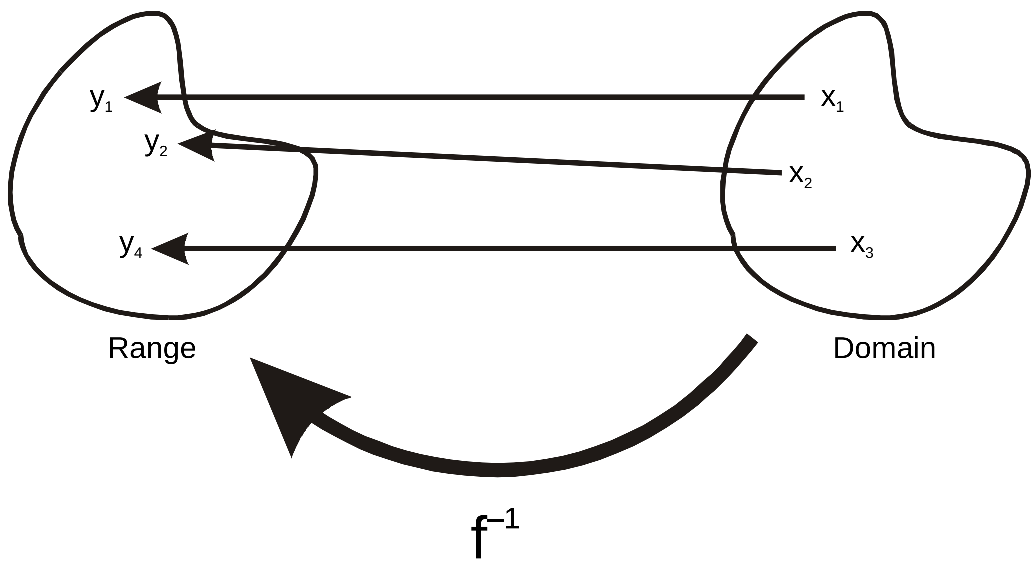
It is obvious from the picture that f^{-1} is not a function. That is, the element, y_1 in the Domain of f^{-1} gets paired with two values in the Range, x_1 and x_3. This pairing violates a fundamental condition of a function. What this tells us is that, while every function has an inverse, not every inverse is a function.

Before continuing a comment needs to be made about the notation $f^{-1}(x)$. Although the –1 exponent generally means reciprocal, when used in function notation it means inverse, not reciprocal. That is, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$.

It is a problem that the inverse of a function is not a function. But this problem is easily fixed by restricting the Domain of $f(x)$. This means that we throw out half of the duplicated points from the Domain of f. In our example we can throw the value x_3 out of the Domain of f. This leaves us

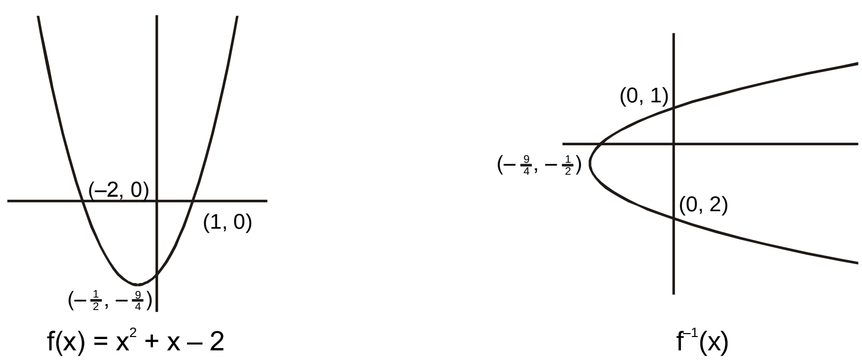


or the points $(x_1, y_1), (x_2, y_2)$, and (x_4, y_3). The inverse function would consist of the points $(y_1, x_1), (y_2, x_2)$, and (y_3, x_4). Now the inverse function is a function as desired. Also, because the Domain variable is x and the Range variable is y we can write

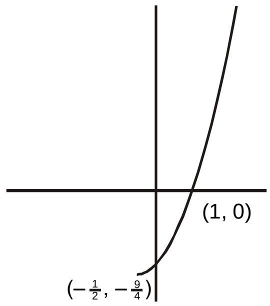


or the points $(x_1, y_1), (x_2, y_2)$, and (x_3, y_4) found by exchanging the x and y variables.

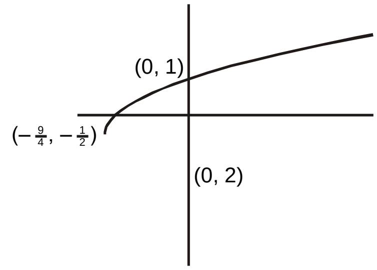
This is the key to finding and graphing the inverse of a function, exchange the Domain and Range or x and y values. For example, graph the function $f(x) = x^2 + x - 2$. Graph the inverse $f^{-1}(x)$ and restrict the domain of $f(x)$ so that the inverse will be a function. The intercepts are (–2, 0) and (1, 0) and the vertex is at $\left(-\frac{1}{2}, -\frac{9}{4}\right)$. Swapping the x and y coordinates (exchanging the Domain and Range values) we get the points (0, –2), (0, 1) and $\left(-\frac{9}{4}, -\frac{1}{2}\right)$. Graphing these sets of points and corresponding graphs we get:



The function $f(x)$ is not 1-1. As a result the inverse is not a function. We need to restrict the Domain of $f(x)$ so that its inverse is a function. The vertex of $f(x)$ is at $\left(-\frac{1}{2}, -\frac{9}{4}\right)$. It's at this point where the left and right sides of the graph reflect. So this is the point where we want to restrict the Domain. If we eliminate the points on the left of the vertex (we could just as easily eliminate the points on the right) we also eliminate the corresponding points on the inverse, or the points below the vertex of the inverse. This gives us the graphs:

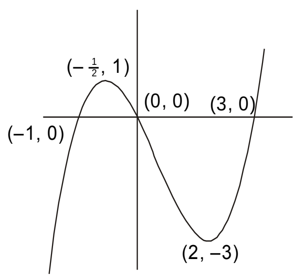


f(x) = x² + x – 2
Domain restricted
to x ≥ – ½



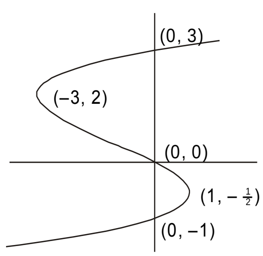
$f^{-1}(x)$
Domain of f(x)
restricted to x ≥ – ½

Example 1: Given the graph of the function $f(x)$ below, graph the inverse of the function and restrict the Domain of f so that the inverse will be a function.



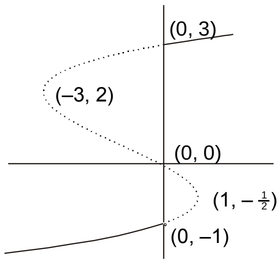
f(x)

Exchanging the Domain and Range values (swapping the x's and y's) we get the following graph:

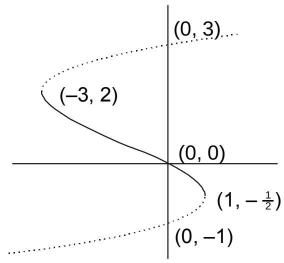


$f^{-1}(x)$

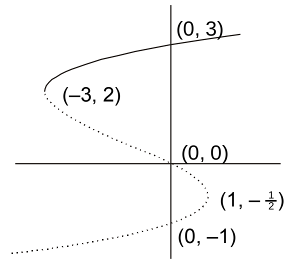
Again, the inverse of $f(x)$ is not a function. So we need to restrict the Domain of f so that f^{-1} is a function. There are several ways of doing this. Three possibilities are given below.



$f^{-1}(x)$
Domain of f(x) restricted to x < –1 or x ≥ 3.

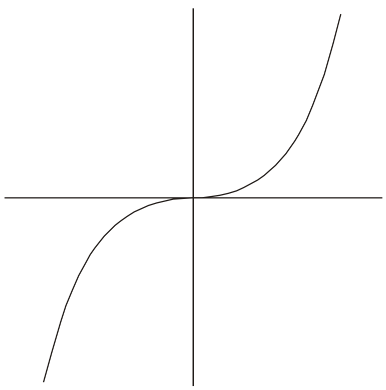


$f^{-1}(x)$
Domain of f(x) restricted to –1 ≤ x ≤ 3.

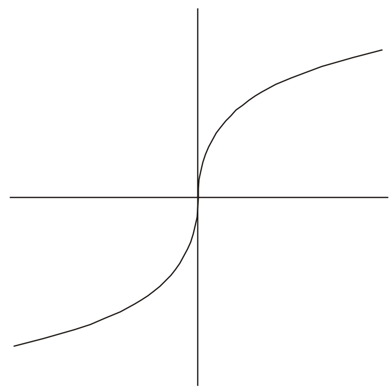


$f^{-1}(x)$
Domain of f(x) restricted to x ≥ 2.

If the function $f(x)$ is 1-1 then there is no need to restrict the Domain because the inverse will automatically be a function. Consider the function $f(x) = x^3$ with inverse function $f(x) = \sqrt[3]{x}$ shown below.



$f(x) = x^3$



$f(x) = \sqrt[3]{x}$

Swapping the Domain and Range values of a function also allows us to find the equation of the inverse.

Example 2: Find the inverse of the function.

$f(x) = (x+2)^3 - 4$	"$f(x) =$" is the same as "$y =$"
$y = (x+2)^3 - 4$	To find the inverse we swap the x and y variables
$x = (y+2)^3 - 4$	Solve for y, first add 4
$x + 4 = (y+2)^3$	Cube root both sides
$\sqrt[3]{x+4} = y + 2$	Subtract 2 from both sides
$\sqrt[3]{x+4} - 2 = y$	Here, y represents the inverse function
$f^{-1}(x) = \sqrt[3]{x+4} - 2$	Final answer

Example 3: Find the inverse of the function

$f(x) = (x-3)^2 + 6$	"$f(x) =$" is the same as "$y =$"
$y = (x-3)^2 + 6$	Swap the x and y variables
$x = (y-3)^2 + 6$	Solve for y, first subtract 6
$x - 6 = (y-3)^2$	Square root both sides
$\pm\sqrt{x-6} = y - 3$	Add 3 to both sides
$3 \pm \sqrt{x-6} = y$	The ± comes from taking the square root. $f(x)$ is not a 1-1 function. The vertex of the function is at the point (3, 6) Restricting the Domain to values of $x \geq 3$
$f^{-1}(x) = 3 + \sqrt{x-6}$ where the domain of $f(x)$ is restricted to $x \geq 3$	Final answer

One useful property of the inverse function is that $f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$. This is because, regardless which side of the function we start on (Domain or range), when we apply the function rules in the proper order we always end up where we started. Back at x.

2.3 Inverse Functions Practice

Find the inverse function for each of the following functions

1. $p(t) = (t-2)^2 + 3$
2. $z(x) = (x-2)^3 + 3$
3. $h(x) = \sqrt{x-2} + 3$
4. $q(p) = \dfrac{p-2}{p+3}$
5. $A(b) = \dfrac{2b+1}{b-2}$
6. $f(x) = 2\sqrt{x+1} + 4$
7. $f(x) = \dfrac{3x-5}{2}$
8. $z(r) = (2r-3)^2 - 6$
9. $f(x) = (4x-2)^2 - 1$
10. $f(x) = \sqrt[3]{3-x} + 8$
11. $f(x) = \dfrac{2}{x+3} - 1$
12. $f(x) = 3 - \dfrac{1}{x-2}$

13. Graph $f(x) = x^2 + 1$ and its inverse. Restrict the domain of $f(x)$ so that $f^{-1}(x)$ is a function.

14. Graph $f(x) = x^3 + 1$ and its inverse. Restrict the domain of $f(x)$ so that $f^{-1}(x)$ is a function.

15. Graph $f(x) = x^3 - 1$ and its inverse. Restrict the domain of $f(x)$ so that $f^{-1}(x)$ is a function.

16. Graph $f(x) = |x^3 - 1|$ and its inverse. Restrict the domain of $f(x)$ so that $f^{-1}(x)$ is a function.

17. Which of the following functions are 1-1? For each of the functions find the inverse.

a. $f(x) = x + 4$

b. $f(x) = 2x$

c. $f(x) = \dfrac{4}{x+7}$

d. $f(x) = \dfrac{x+4}{x-3}$

e. $f(x) = x^3 - 1$

f. $f(x) = x^4 - 1$

g. $f(x) = (x-2)^2 + 1$

h. $f(x) = \sqrt{x}$

i. $f(x) = \sqrt[3]{x}$

j. $f(x) = \sqrt{2x+3}$

k. $f(x) = \sqrt[3]{2x+3}$

l. $f(x) = 5$

m. $f(x) = x^2 - 2x + 2$

n. $f(x) = 3x^2 - 6x + 1$

18. Show that each of the following functions are inverses by showing that $f\left(g(x)\right) = x$

a. $f(x) = x^2 - 4;\ \ g(x) = \sqrt{x+4}$

b. $f(x) = \dfrac{1}{x-1};\ \ g(x) = \dfrac{1}{x} + 1$

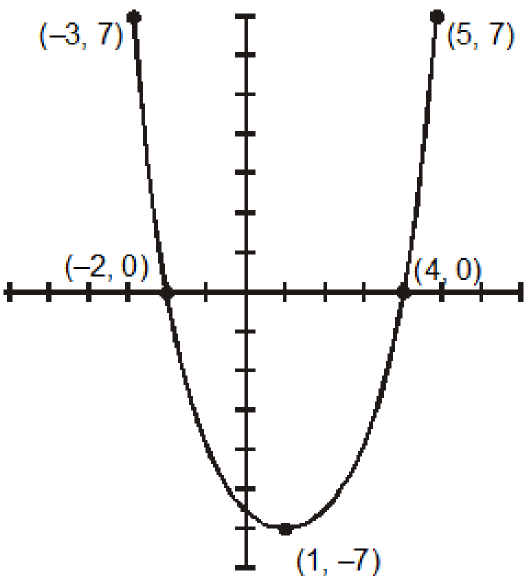
c. $f(x) = 2x + 3;\ \ g(x) = \dfrac{x-3}{2}$

d. $f(x) = \dfrac{2x+1}{2x-1};\ \ g(x) = \dfrac{x+1}{2(x-1)}$

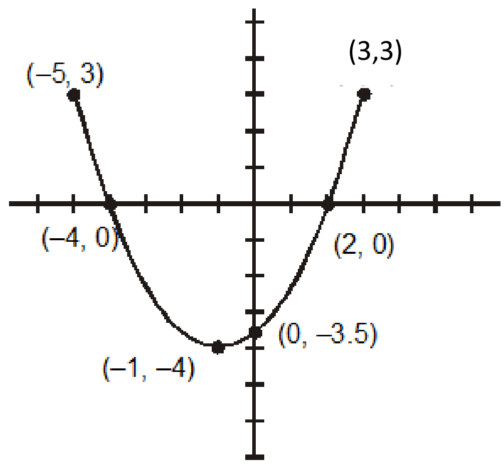
19. What conditions must be placed on $a, b, c,$ and d in $f(x) = \frac{ax+b}{cx+d}$ so that $f^{-1}(x) = f(x)$?

20. Graph the inverse of each of the following functions. Where the function is not 1-1 restrict the domain of the function so that the inverse will be a function.

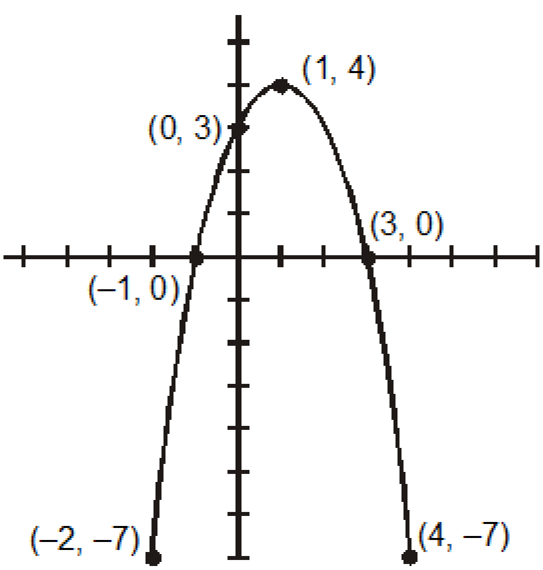
a.



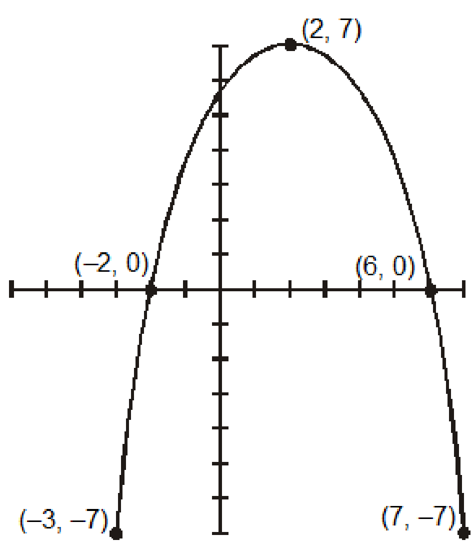
b.



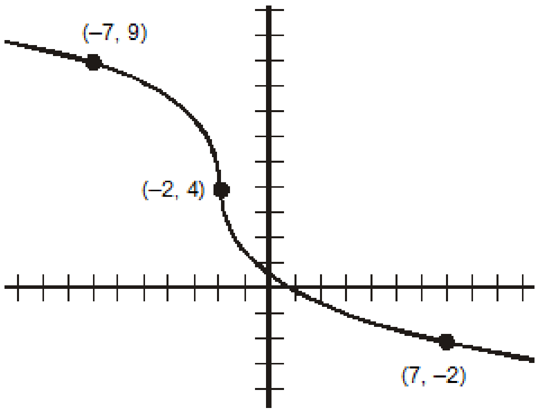
c.



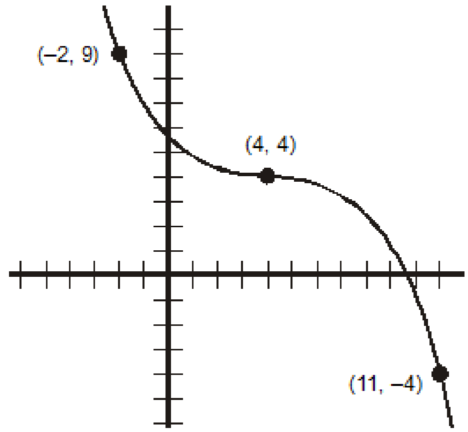
d.



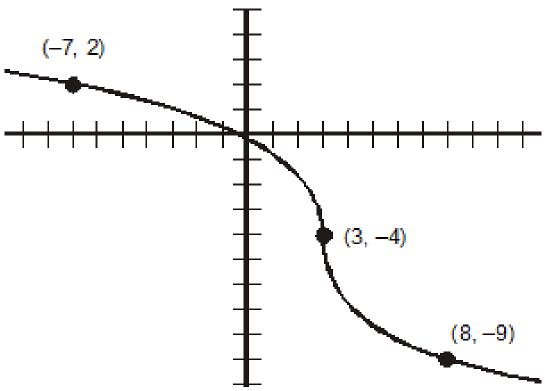
e.


(–7, 9)
(–2, 4)
(7, –2)

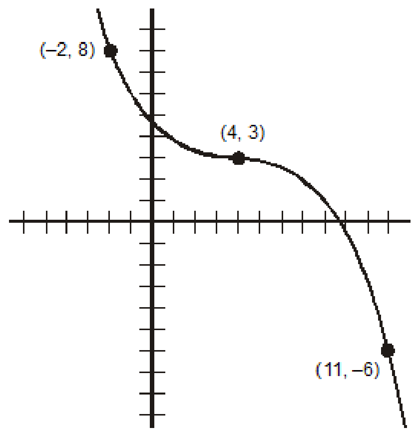
f.


(–2, 9)
(4, 4)
(11, –4)

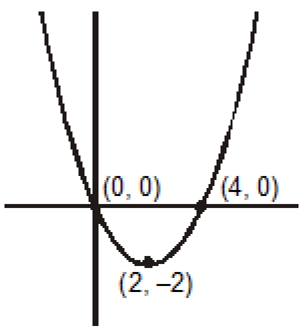
g.


(–7, 2)
(3, –4)
(8, –9)

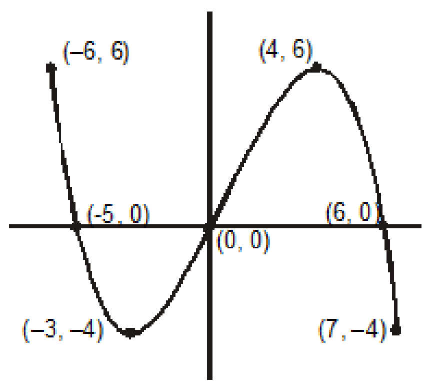
h.


(–2, 8)
(4, 3)
(11, –6)

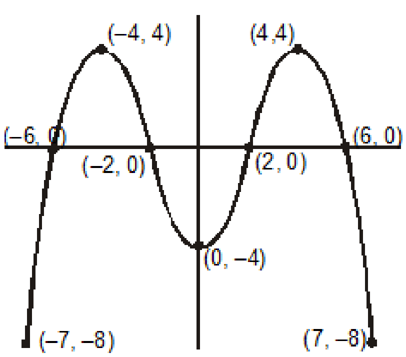
i.


(0, 0)
(4, 0)
(2, –2)

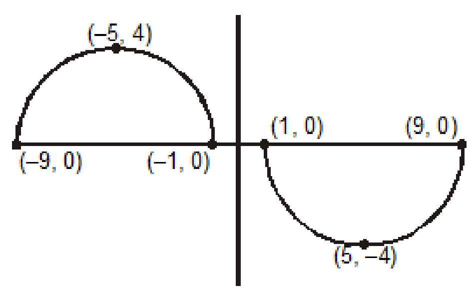
j.


(–6, 6)
(4, 6)
(-5, 0)
(6, 0)
(0, 0)
(–3, –4)
(7, –4)

k.

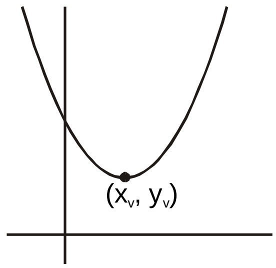

(–4, 4)
(4,4)
(–6, 0)
(6, 0)
(–2, 0)
(2, 0)
(0, –4)
(–7, –8)
(7, –8)

l.

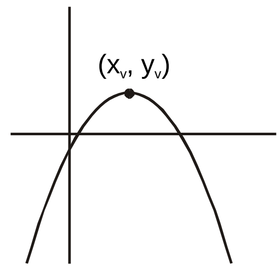

(–5, 4)
(1, 0)
(9, 0)
(–9, 0)
(–1, 0)
(5, –4)

2.4 Applications of Functions

Consider the quadratic equation $y = f(x) = ax^2 + bx + c$. We know from previous work that there are two possible graphs of a quadratic depending upon whether the leading coefficient is positive or negative. These are shown below.



a > 0



a < 0

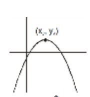
If the leading coefficient is positive, $a > 0$, then the tail is positive and the vertex (x_v, y_v) is a minimum. And, if the leading coefficient is negative, $a < 0$, then the tail is negative and the vertex (x_v, y_v) is a maximum.

We also know that the x-coordinate of the vertex, x_v is found by the formula $x_v = -\frac{b}{2a}$ and that the y-coordinate, $y_v = f(x_v)$. What we want to do in this section is use the vertex to maximize (find the largest value) or minimize (find the smallest value) a function. This maximum or minimum value is the y coordinate of the vertex, y_v. In mathematical terminology, the function that we are maximizing or minimizing is called the objective function and the maximum or minimum value is called an extrema of the function.

Example 1: Find the extrema of the function $f(x) = -2x^2 + 4x + 6$.

$$f(x) = -2x^2 + 4x + 6$$

the Extremum of the function is a maximum because the leading coefficient is negative



Find the x_v from formula $x_v = -\frac{b}{2a}$

$$x_v = -\frac{4}{2(-2)} = \frac{-4}{-4} = 1$$

The maximum value of the function, $y_v = f(x_v)$

$$f(1) = -2(1)^2 + 4(1) + 6$$

Simplify exponents

$$f(1) = -2(1) + 4(1) + 6$$

Multiply

$$f(1) = -2 + 4 + 6$$

Add

$$f(1) = 8$$

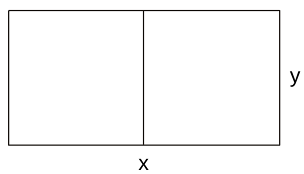
Final answer

If there are multiple variables we need multiple equations to reduce our objective function down to two variables, y and x. These extra equations are called constraint equations because they constrain the possible values of our variables.

Example 2: Find two numbers whose sum is 12 and whose product is a maximum.

x, y	Equation for product to maximize
$P = xy$	Constraint: sum is 12
$x + y = 12$	Solve for y
$y = 12 - x$	Substitute into objective equation
$P = x(12 - x)$	Multiply
$P = -x^2 + 12x$	Find location of maximum from $x_v = -\frac{b}{2a}$
$x_v = \frac{-12}{2(-1)} = \frac{-12}{-2} = 6$	Find other variable
$y = 12 - (6) = 6$	Find product
$P = (6)(6) = 36$	Under the given constraint, this is the largest possible value the product can have

Example 3: A farmer has 120 feet of fence to build a double corral. To get the corral he will fence a rectangular region with a fence segment dividing the rectangle down the middle as shown in the diagram. What dimensions should he use if he wants to maximize the area of the corral? What is the maximum area?

Dimensions: x, y	Equation for area
$A = xy$	Draw picture
	Note two sides of x and three sides of y. Constraint: total fence is 120 feet
$2x + 3y = 120$	Solve for y
$y = -\frac{2}{3}x + 40$	Substitute into objective equation
$A = x\left(-\frac{2}{3}x + 40\right)$	Multiply
$A = -\frac{2}{3}x^2 + 40x$	Find location of maximum from $x_v = -\frac{b}{2a}$
$x_v = \frac{-40}{2\left(\frac{-2}{3}\right)} = \frac{-40}{\frac{-4}{3}} = -40 \cdot \frac{-3}{4} = 30$	Find other variable
$y = -\frac{2}{3}(30) + 40$ $y = -2(10) + 40$ $y = -20 + 40$ $y = 20$	Find the area
$A = (30)(20) = 600$	Clearly state solution in terms of problem
The fence will be 30 ft by 20 ft with an area of 600 ft^2	Final answer

2.4 Applications of Functions Practice

1. Two numbers add to 5. What is the largest possible value of their product?

2. Find two numbers adding to 20 such that the sum of their squares is as small as possible.

3. The difference of two numbers is 1. What is the smallest possible value for the sum of their squares?

4. For each quadratic function specified below, state whether it would make sense to look for a highest or a lowest point on the graph. Then determine the coordinates of that point.

a. $y = 2x^2 - 8x + 1$
b. $y = -3x^2 - 4x - 9$
c. $h = -16t^2 + 256t$
d. $f(x) = 1 - (x + 1)^2$
e. $g(t) = t^2 + 1$
f. $f(x) = 1000x^2 - x + 100$

5. Among all rectangles having a perimeter of 25 m, find the dimensions of the one with the largest area.

6. What is the largest possible area for a rectangle whose perimeter is 80 cm?

7. What is the largest possible area for a right triangle in which the sum of the lengths of the two shorter sides is 100 in?

8. The perimeter of a rectangle is 12 m. Find the dimensions for which the diagonal is as short as possible.

9.
a. Minimize $S = 6x^2 - 2xy + 5y^2$ given that $x + y = 13$.
b. Minimize $S = 12x^2 + 4xy - 10y^2$ given that $x + y = 14$.
c. Maximize $S = 3x^2 + 5xy - 2y^2$ given that $x + y = 8$.
d. Maximize $S = 4x^2 + 3xy - 5y^2$ given that $x + y = -8$.
e. Maximize $S = -2x^2 + 3xy - 5y^2$ given that $x + y = 20$.
f. Minimize $S = 3x^2 + 2xy + 2y^2$ given that $3x - 2y = 42$.
g. Minimize $S = 2x^2 + 5xy - y^2$ given that $4x - y = 12$.
h. Minimize $S = 3x^2 - xy + 2y^2$ given that $x - 2y = 24$.
i. Maximize $S = -3x^2 + 2xy + 4y^2$ given that $2x - 5y = -9$.
j. Maximize $S = -x^2 + 6xy - 7y^2$ given that $2x - 3y = 2$.

10. Two numbers add to 6.

a. Let T denote the sum of the squares of the two numbers. What is the smallest possible value for T?
b. Let S denote the sum of the first number and the square of the second. What is the smallest possible value for S?
c. Let U denote the sum of the first number and twice the square of the second number. What is the smallest possible value for U?
d. Let V denote the sum of the first number and the square of twice the second. What is the smallest possible value for V?

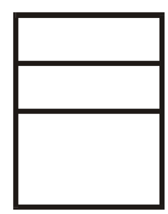
11. Suppose that the height of an object shot straight up is given by $h(t) = 512t - 16t^2$ (h in feet and t in seconds). Find the maximum height and the time at which the object hits the ground.

12. A baseball is thrown straight up, and its height as a function of time is given by the formula $h(t) = -16t^2 + 32t$ (h in feet and t in seconds).

a. Find the height of the ball when $t = 1$ and when $t = \frac{3}{2}$.
b. Find the maximum height of the ball and the time at which that height is attained.
c. At what time(s) is the height 7 feet?

13. a. What number exceeds its square by the greatest amount?
b. What number exceeds twice its square by the greatest amount?

14. Suppose that you have 1800 meters of fencing available with which to build three adjacent rectangular corrals as shown in the figure. Find the dimensions so that the total enclosed area is as large as possible.


Problem 17

15. Five hundred feet of fencing is available for a rectangular pasture alongside a river, the river serving as one side of the rectangle (so only three sides require fencing). Find the dimensions yielding the greatest area.

16. Let $A = 3x^2 + 4x - 5$ and $B = x^2 - 4x - 1$. Find the minimum value of $A - B$.

17. Let $R = 0.4x^2 + 10x + 5$ and $C = 0.5x^2 + 2x + 101$. For which value of x is $R - C$ a maximum?

18. Suppose that the revenue generated by selling x units of a certain commodity is given by $R = -\frac{1}{5}x^2 + 200x$. Assume that R is in dollars. What is the maximum revenue possible in this situation?

19. Suppose that the function $p(x) = -\frac{1}{4}x + 30$ relates the selling price p of an item to the quantity sold, x. Assume P is in dollars. For which value of x will the corresponding revenue be a maximum? What is this maximum revenue and what is the unit price in this case?

20. A piece of wire 200 cm long is to be cut into two pieces of lengths x and $200 - x$. The first piece is to be bent into a circle and the second piece into a square. For which value of x is the combined area of the circle and square as small as possible?

21. A 30 in piece of string is to be cut into two pieces. The first piece will be formed into the shape of an equilateral triangle and the second piece into a square. Find the length of the first piece if the combined area of the triangle and the square is to be as small as possible?

22. a. Same as exercise 23 except both pieces are to be formed into squares.
b. Could you have guessed the answer to part a?

23. The action of sunlight on automobile exhaust produces air pollutants known as photochemical oxidants. In a study of cross-country runners in Los Angeles, it was shown that running performances can be adversely affected when the oxidant level reaches 0.03 parts per million. Let us suppose that on a given day the oxidant level L is approximated by the formula

$$L = 0.059t^2 - 0.354t + 0.557 \quad (0 \leq t \leq 7)$$

Here, t is measured in hours, with $t = 0$ corresponding to 12 noon, and L is in parts per million. At what time is the oxidant level L a minimum? At this time, is the oxidant level high enough to affect a runner's performance?

24. If $x + y = 1$, find the largest possible value of the quantity $x^2 - 2y^2$.

25. Find the smallest possible value of the quantity $x^2 + y^2$ under the restriction that

$$2x + 3y = 6$$

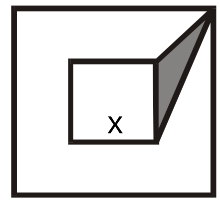
26. Through a type of chemical reaction known as autocatalysys, the human body produces the enzyme trypsin from the enzyme trypsinogen. (Trypsin then breaks down proteins into amino acids, which the body needs for growth.) Let r denote the rate of this chemical reaction in which trypsin is formed from trypsinogen. It has been shown experimentally that $r = kx(a - x)$, where r is the rate of the reaction, k is a positive constant, a is the initial amount of trypsinogen, and x is the amount of trypsin produced (so x increases as the reaction proceeds). Show that the reaction rate r is a maximum when $x = \frac{a}{2}$. In other words, the speed of the reaction is the greatest when the amount of trypsin formed is half of the original amount of trypsinogen.

27. a. Let $x + y = 15$. Find the minimum value of the quantity $x^2 + y^2$.

b. Let C be a constant and $x + y = C$. Show that the minimum value of $x^2 + y^2$ is $\frac{C^2}{2}$. Then use this result to check your answer in part a.

28. Suppose that A, B, and C are positive constants and that $x + y = C$. Show that the minimum value of $Ax^2 + By^2$ occurs when $x = \frac{BC}{A+B}$ and $y = \frac{AC}{A+B}$.

29. The figure at the right shows two concentric squares. The side of the outside square is 1 unit. For which value of x is the shaded area a maximum?



Problem 32

30. Find the largest value of the function $f(x) = \frac{1}{x^4 - 2x^2 + 1}$.

31. A rancher, who wishes to fence off a rectangular area, finds that the fencing in the east-west direction will require extra reinforcement due to the strong prevailing winds. Because of this, the cost of fencing in the east-west direction will be \$12 per linear yard, as opposed to a cost of \$8 per yard for fencing in the north-south direction. Find the dimensions of the largest possible rectangular area that can be fenced for \$4800.

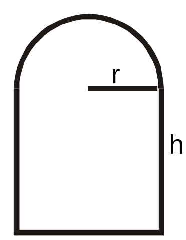
32. Let $f(x) = (x-a)^2 + (x-b)^2 + (x-c)^2$, where a, b, and c are constants. Show that $f(x)$ will be a minimum when x is the average of a, b, and c.

33. Let $y = a_1(x - x_1)^2 + a_2(x - x_2)^2$, where a_1, a_2, x_1, and x_2 are all constants. Further, suppose that a_1 and a_2 are both positive. Show that the minimum of this function occurs when $x = \frac{a_1x_1 + a_2x_2}{a_1 + a_2}$.

34. Among all rectangles with a given perimeter P, find the dimensions of the one with the shortest diagonal.

35. Show that the maximum possible area for a rectangle inscribed in a circle of radius R is $2R^2$.

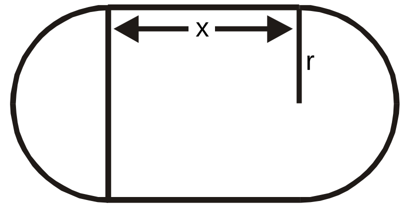
36. A Norman window is in the shape of a rectangle surmounted by a semicircle, as shown in the figure. Assume that the perimeter of the window is P, a constant. Find the values of h and r when the area is a maximum and find this area.



Problem 43

37. A rectangle is inscribed in a semicircle of radius 3. Find the largest possible area for this rectangle.

38. An athletic field with a perimeter of $\frac{1}{4}$ mi consists of a rectangle with a semicircle at each end, as shown in the figure below. Find the dimensions x and r that yield the greatest possible area for the rectangular region.



Problem 45

39. Find the values of x that make y a minimum or a maximum, as the case may be. Find the corresponding y value and indicate whether it is a minimum or a maximum.

a. $y = 3x^4 - 12x^2 - 5$
b. $y = \sqrt{4x - x^2}$

40. By analyzing sales figures, the economist for a stereo manufacturer knows that 150 units of a top of the line turntable can be sold each month when the price is set at $p = \$200$ per unit. The figures also show that for each \$10 hike in price, 5 fewer units are sold each month.

a. Let x denote the number of units sold per month and let p denote the price per unit. Find a linear function relating p and x.
b. Express the revenue R as a function of x.
c. What is the maximum revenue? At what level should the price be set to achieve this maximum revenue?

41. Imagine that you own an orchard of orange trees. Suppose from past experience you know that when 100 trees are planted, each tree will yield approximately 240 oranges. Furthermore, you've noticed that when additional trees are planted, the yield per (each) tree in the orchard decreases. Specifically, you have noted that the yield per tree decreases by about 20 oranges for each additional tree planted. Approximately how many trees should be planted in the orchard to produce the largest possible total yield of oranges?

42. An appliance firm is marketing a new refrigerator. It determines that in order to sell x refrigerators, its price per refrigerator must be $p = D(x) = 280 - 0.4x$. It also determines that its total cost of producing x refrigerators is given by $C(x) = 5000 - 0.6x^2$.

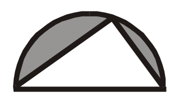
a. How many refrigerators must the company produce and sell in order to maximize profit?
b. What is the maximum profit?
c. What price per refrigerator must be charged in order to make this maximum profit?

43. The owner of a 30 unit motel find that all units are occupied when the charge is \$20 per day per unit. For every increase of x dollars in the daily rate, there are x units vacant. Each occupied room costs \$2 per day to service and maintain. What should he charge per unit per day in order to maximize profit?

44. A university is trying to determine what price to charge for football tickets. At a price of \$6 per ticket, it averages \$70,000 people per game. For every increase of \$1, it loses 10,000 people from the average number. Every person at the game spends an average of \$1.50 on concessions. What price per ticket should be charged in order to maximize revenue? How many people will attend at that price?

45. When a theater owner charges \$3 for admission, there is an average attendance of 100 people. For every 10 cent increase in admission, there is a loss of 1 customer from the average. What admission should be charged in order to maximize revenue?

46. An apple farm yields an average of 30 bushels of apples per tree when 20 trees are planted on an acre of ground. Each time 1 more tree is planted per acre, the yield decreases 1 bushel per tree due to the extra congestion. How many trees should be planted in order to get the highest yield?


Problem 54

47. A triangle is removed from a semicircle of radius R as shown in the figure. Find the area of the remaining portion of the circle if it is to be a minimum.

48. Let $f(x) = x^2 + px + q$, and suppose that the minimum value of this function is 0. Show that $q = \frac{p^2}{4}$.

49. Suppose that x and y are both positive numbers and that their sum is 4. Find the smallest possible value for the quantity $\frac{1}{xy}$.

2.5 Reading Graphs of Functions

When we discussed functions we stated that functions are rules that pair off elements of the domain with elements of the range. These pairings may be written in the form of an ordered pair (x, y) where x is an element of the domain and y is an element of the range.

As you will no doubt remember from your algebra, ordered pairs are also used to plot points on a Cartesian coordinate system. This analogy gives us a natural way to display the points of a function. By letting the x axis represent the domain and the y axis represent the range we can plot all of the points (ordered pairs) which comprise the function.

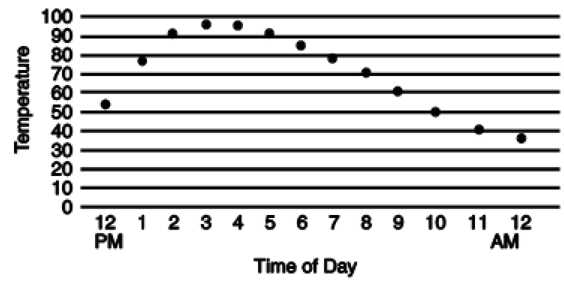


Figure 1

Graphing a function in this manner can be very helpful in recognizing various properties of the function. Suppose that we made a record of the temperature at various times during the day (See figure 1) with time as the domain and temperature as the range.

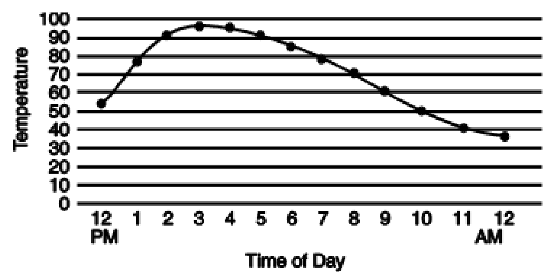


Figure 2

From the graph of the temperature plots we can see features of the function that would not be obvious from a simple listing of the points. For example, the temperature rises fastest during the period between 1 and 2. It reaches its peak between 3 and 4. The drop in temperature is fastest between 11 and 12.

Because temperature changes in a continuous fashion we can logically assume that the intermediate values which were not recorded fill in the graph forming a continuous curve as shown in figure 2. We also know that the relationship between time and temperature must be a function. It is not possible to have two different temperatures in the same place at the same time.

How to Read a Graph

It is important to be able to read the properties of a function from its graph. The types of features you should be able to recognize are domain and range of a function, function values, maximum and minimum values of a function, positive and negative range values, asymptotes and limits at infinity.

A point on the x axis is in the domain of a function if the graph of the function passes over (or under) that point. An easy way to tell is to draw a vertical line through the point on the x axis and if the vertical line crosses the graph then the x value is in the domain of the function.

Similarly, a point on the y axis is in the range of a function if the graph of the function passes the point on either the left or the right. Here, a horizontal line drawn through the point on the y axis must pass through the graph.

In both of the above cases, the graph of the function could pass directly through the respective axis. Obviously a point would belong to the domain of a function if the graph goes through the x axis, or the range if the graph goes through the y axis at that point.

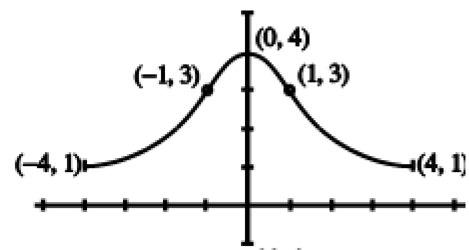
Vertical Line Test

If the point (x_1, y_1) is on the graph of a function f, then, under the function rule $y_1 = f(x_1)$. Thus, the function evaluated at x_1 is equal to y_1.

This gives us a quick method to see if a graph is the graph of a function. Each domain element of a function can only have one range element corresponding to it. If a vertical line should cross the graph at 2 or more points then the graph cannot be the graph of a function. This is called the vertical line test for a function.

Example 1: Consider the graph of the function f in example 1.

a. What is the domain of f?
b. What is the range of f?
c. What is $f(1)$?
d. What domain values of f give $f(x) = 1$?



y = f(x)
Example 1

This is the graph of a function. Any vertical line which crosses the graph crosses at only one point.

Solutions to Example 1:

a. The domain of f is all points on the x axis that have a corresponding point on the graph. In this case only those x values between –4 and 4 or the interval $-4 \leq x \leq 4$ are inside the domain. Note that any vertical line drawn through the x axis outside this interval does not cross the graph. Any vertical line drawn inside the interval crosses the graph.

b. The range of f is all points on the y axis that have a corresponding point on the graph. Here the range is the interval $1 \leq y \leq 4$. Every horizontal line drawn inside this interval crosses the graph. Outside this interval horizontal lines do not cross the graph.

c. $f(1)$ is that y coordinate on the graph that has 1 for the corresponding x coordinate. Looking at the graph we see that the point (1,3) lies on the graph of f. Consequently, the range value corresponding to a domain value of 1 is 3, or $f(1) = 3$.

d. Here we need to find those domain values that return a value of 1. The graph of f has two such points, one at (–4,1) and the other at (4,1). Therefore, we have 2 domain values that return a range value of 1. These are –4 and 4. Consequently, $f(-4) = 1$ and $f(4) = 1$.

Positive and Negative Values

The y coordinates on the graph represent the range values of the function. Whenever the graph of the function is above the x axis the y coordinates are positive and the corresponding function values are positive. If the graph lies below the x axis the y coordinates are negative and the corresponding function values are negative. In the above example (example 1), all graph points are above the x axis. Therefore, $f(x) > 0$ for each domain element.

Vertical Asymptotes

Vertical asymptotes are vertical lines that the graph of a function approach but never cross.

Figure 3 is the graph of the function $f(x) = \frac{1}{x}$, $x = 0$ is not a point in the domain of f because the function is undefined at this point. On the graph we see that the function moves up or down approaching the line $x = 0$ (the y axis) but does not cross it.

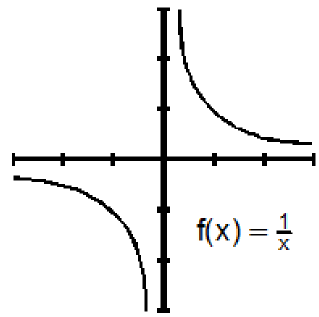


Figure 3

The y axis is a vertical asymptote of the function f. Whenever we have a vertical asymptote it passes through a point not in the domain of the given function.

Limits at Infinity

Vertical asymptotes are not the only asymptotes that a graph might have. When the domain values get large in either the positive or negative direction it may happen that the graph will approach the curve of some other function. Looking again at figure 3 we see that the graph of f approaches the x axis (i.e. the y coordinates approach 0) as the domain values get infinitely large.

In mathematical notation we use an arrow $\rightarrow$ to indicate that a value (for either x or y) is approaching some given point. If we were to choose values of x that are getting closer to the point $x = 2$ then we could write $x \rightarrow 2$. If we want to consider values of x that are getting infinitely large we would write $x \rightarrow \infty$. This means that the values of x are moving infinitely far to the right on the number line. If the points were moving infinitely far to the left (infinitely far in the negative direction) then we would write $x \rightarrow -\infty$.

Similarly, if we wanted to indicate that the y values are approaching a point $y = 1$ we would write $y \rightarrow 1$.

In this case the horizontal asymptote $y = 0$ defines a y value that is not in the range of the function. However, this is not always the case. Unlike the vertical asymptote, it is possible for the graph of the function to cross the horizontal asymptote.

In figure 4 we have the graph of a function with two vertical asymptotes and a single horizontal asymptote. The two vertical asymptotes are $x = 3$ and $x = -3$. At these two points the graph shoots off to either + or – infinity. As we approach $x = -3$ from the left side of the graph the function shoots up to $+\infty$. As we approach $x = -3$ from the right side, the graph goes down to $-\infty$. The same thing happens as we approach $x = 3$. Notice that a vertical line drawn through $x = 3$ will not cross the graph and that +3 and –3 are not in the domain of the function.

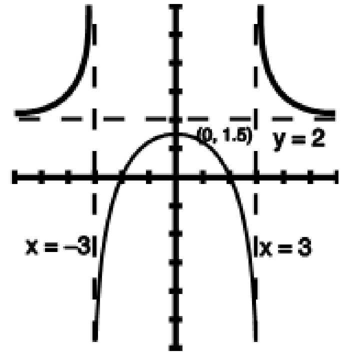


Figure 4

The horizontal asymptote of this function is the line $y = 2$. As we pick larger and larger domain values approaching ∞ the range values tend to get closer to this line. Similarly, as we pick domain values approaching $-\infty$, the graph again approaches the line $y = 2$. The range of this function is all real numbers $y > 2$ and all real numbers $y \leq 1.5$. Any horizontal line drawn between these two values will not cross the graph and, consequently, will not be in the range of the function.

Consider the graph of the function $y = h(x)$ in figure 5. This graph has one vertical asymptote, one horizontal asymptote and another asymptote along the left half of the curve $y = x^2$. The vertical asymptote occurs at the line $x = 4$. Because a function will never cross a vertical asymptote, the point $x = 4$ is not in the domain of h.

The horizontal asymptote occurs at the line $y = 3$. The graph of the function crosses this line at approximately $x = -2$. The line $y = 3$ is a horizontal asymptote as $x \to \infty$, or $h(x) \to 3$ as $x \to \infty$.

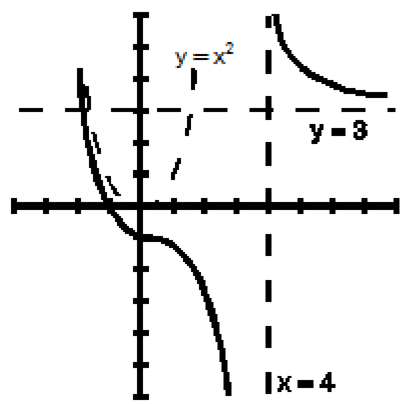


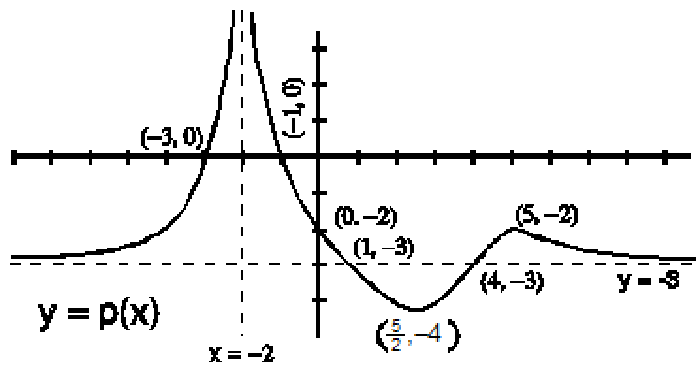
Figure 5

As $x \to -\infty$ we have a different situation. Here the graph of the function approaches a curve rather than a straight line. In particular, the graph values approach the curve $y = x^2$. As a result of this, the graph crosses the horizontal asymptote $y = 3$ at approximately (2, 3). Even though we never cross a vertical asymptote it does not necessarily follow that we never cross any other asymptotes. The horizontal asymptote is an indication of what the range values do as we pick larger and larger domain values. It in no way indicates what happens to the function at any other points.

Other features of this graph are:
1. The domain of h is all real numbers except $x = 4$.
2. The range of h consists of all real numbers (because the graph crosses the line $y = 3$ the point 3 is in the range).
3. It has an x intercept at –1, i.e. $h(-1) = 0$.
4. It has a y intercept at –1, i.e. $h(0) = -1$.

Example 2: Given the following graph of $y = p(x)$,

a. What is the domain of p?
b. What is the range of p?
c. Over what interval(s) is $p(x)$ positive?
d. Over what interval(s) is $p(x)$ negative?
e. For $x \geq \frac{5}{2}$, what is the maximum value(s) of $p(x)$?
f. For $x \geq 1$, what is the minimum value(s) of $p(x)$?
g. What is $p(0)$?
h. For what value(s) of x does $p(x) = 0$?
i. What is $p(5)$?
j. For what value(s) of x does $p(x) = -3$?
k. As $x \to -2$, what does $p(x) \to$?
l. As $x \to +\infty$, what does $p(x) \to$?



Example 2

Solutions to Example 2:

a. The domain of p consists of all x values through which we can draw a vertical line that crosses the graph of p. From the graph, the only vertical line that we can draw that does not cross the graph is through the point $x = -2$. Consequently, the domain of p is all x values (real numbers) except $x = -2$.

b. Similarly, the range of p is all y values through which we can draw a horizontal line that crosses the graph. In this particular example, any horizontal line drawn below $y = -4$ will not cross the graph. All lines drawn above and including $y = -4$ will cross the graph. The range of p is all y values (real numbers) greater than or equal to –4 ($y \geq -4$).

c. The function $p(x)$ is positive whenever the graph of $p(x)$ lies above the x axis (y coordinates are positive). This occurs only in the interval (–3, –1). However, the function p is undefined at $x = -2$ (not an element of the domain of p). $p(x) > 0$ on the intervals $-3 < x < -2$ and $-2 < x < -1$.

d. The graph of p is below the x axis and defined over all other intervals. Therefore, $p(x) < 0$ over the intervals $-\infty < x < -3$ and $-1 < x < \infty$.

e. The maximum value of a function over an interval is the largest y value that the function takes on in the interval. The largest y value is the highest y point on the graph. For the function p over the interval $x \geq \frac{5}{2}$ the highest graph point is (5, –2). The maximum value of the function is $y = -2$.

f. Similarly, the minimum function value is the lowest graph point over the interval. Here the minimum value of the function is $y = -4$ which occurs at the point (2.5, –4).

g. $p(0)$ is the y coordinate corresponding to an x coordinate of $x = 0$. There can only be one y coordinate or the graph would not be the graph of a function. For this graph of $p(x)$, the y coordinate corresponding to $x = 0$ is $y = -2$. Therefore, $p(0) = -2$.

h. When the function values are equal to 0 we are looking at the x intercepts of the graph. Those x coordinate at the x intercepts for p are $x = -3$ and $x = -1$.

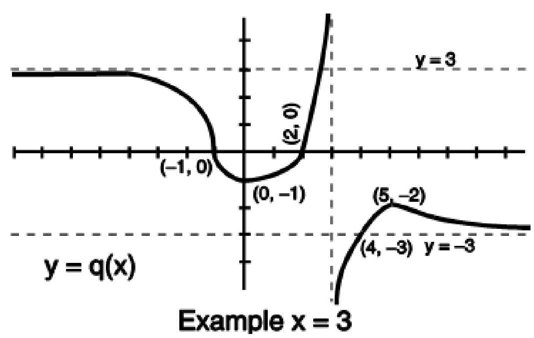
i. To evaluate $p(5)$ we need to find the y coordinate corresponding to $x = 5$. On the graph this point is seen to be $y = -2$. Therefore, $p(5) = -2$.

j. $p(x) = -3$ means $y = -3$. The values of $x = 1$ and 4 are paired to $y = -3$.

k. As the x values approach –2, the y coordinates of the graph get infinitely large in the positive y direction. As a result, $p \to \infty$ as $x \to -2$.

l. As the x values approach $+\infty$, the graph approaches the line $y = -3$. As $x \to -\infty$, the graph again approaches the line $y = -3$. Therefore, as $x \to +\infty, p \to -3$.

Example 3: Given the following graph of $y = q(x)$,



a. What is the domain of q?
b. What is the range of q?
c. Over what interval(s) is q positive?
d. Over what interval(s) is q negative?
e. What is $q(0)$?
f. What domain values give $q(x) = 0$?
g. On the interval $x < 3$, what are the maximum and minimum values of q?
h. On the interval $x > 3$, what are the maximum and minimum values of q?
i. As $x \to 3$, what does q do?
j. As $x \to \pm\infty$, what does q do?

Solutions to Example 3

a. The domain of q is all values of x where we can draw a vertical line that crosses the graph of q. The only x value that does not cross the graph is the line $x = 3$. Therefore, the domain of q is all x's (real numbers) except $x = 3$.

b. The range of q is all values of y where we can draw a horizontal line that crosses the graph of q. We can do this for all y values except the interval $-2 < y < -1$. Therefore, the range of q is all y values (real numbers) except $-2 < y < -1$.

c. q is positive when the graph of q lies above the x axis. The intervals that have q above the x axis are $-\infty < x < -1$ and $2 < x < 3$.

d. The negative values of q are those intervals where the graph lies below the x axis. These intervals are $-1 < x < 2$ and $3 < x < \infty$.

e. $q(0)$ is the y intercept of q. This point is $y = -1$.

f. The domain values of $q(x) = 0$ are the x intercepts of q. These points are $x = -1$ and $x = 2$.

g. On the interval $x < 3$ the minimum value of q is $y = -1$. There is no maximum value because the graph shoots off to ∞.

h. On the interval $x > 3$ the maximum value of q is $y = -2$. There is no minimum value because the graph approaches $-\infty$.

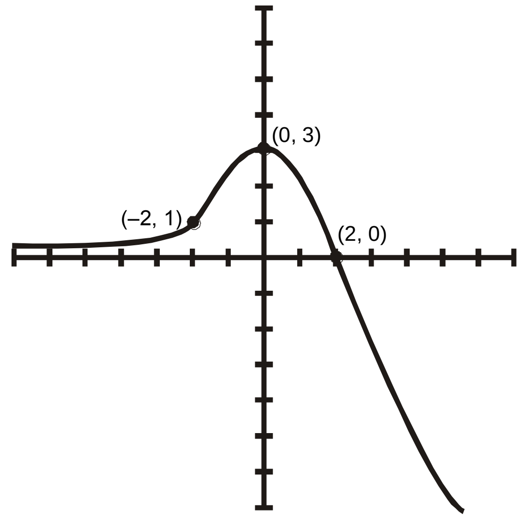
i. As $x \to 3$, the graph goes in two different directions. To differentiate between the two directions we will talk about x approaching 3 from the right or positive side of 3 (which we will write $x \to 3^+$) and x approaching 3 from the left or negative side of 3 (which we will write $x \to 3^-$). As $x \to 3^+$ the graph of q drops off to negative infinity or $q \to -\infty$. As $x \to 3^-$, $q \to +\infty$, or the graph rises to infinity.

j. As $x \to \pm\infty$, again the graph does two different things. As $x \to -\infty$ the graph of q approaches the line $y = 3$. As $x \to +\infty$ the graph of q approaches the line $y = -3$.

2.5 Reading Graphs of Functions Practice

1. Given the graph of $y = f(x)$ at the right,

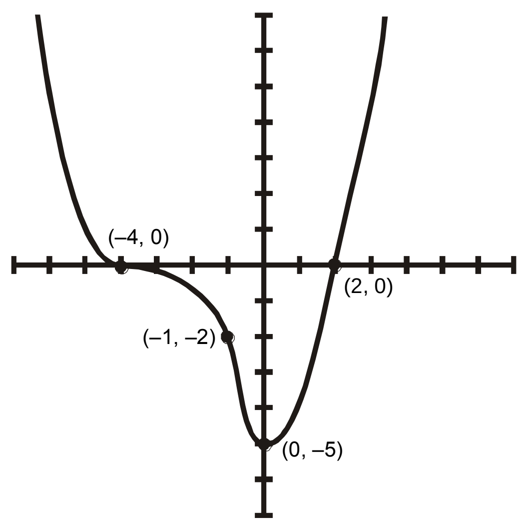
   a. What is the domain of $f(x)$?
   b. What is the range of $f(x)$?
   c. What is $f(0)$?
   d. What value of x give $f(x) = 0$?
   e. What is $f(-2)$?
   f. What is the maximum value of $f(x)$?
   g. Where is $f(x)$ increasing?
   h. Where is $f(x)$ positive?
   i. As $x \to -\infty$, what does f do?
   j. As $x \to \infty$, what does f do?



y = f(x)

2. Given the graph of $y = f(x)$ at the right,

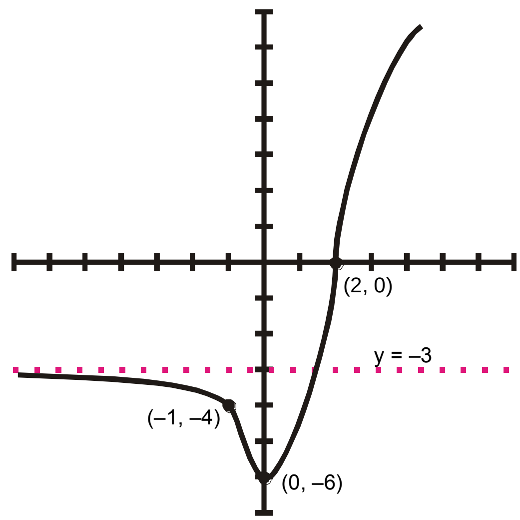
   a. What is the domain of $f(x)$?
   b. What is the range of $f(x)$?
   c. What is $f(0)$?
   d. What values of x give $f(x) = 0$?
   e. What is the minimum value of $f(x)$?
   f. Where is $f(x)$ decreasing?
   g. Where is $f(x)$ negative?
   h. As $x \to -\infty$, what does f do?
   i. As $x \to \infty$, what does f do?



y = f(x)

3. Given the graph of $y = f(x)$ at the right,

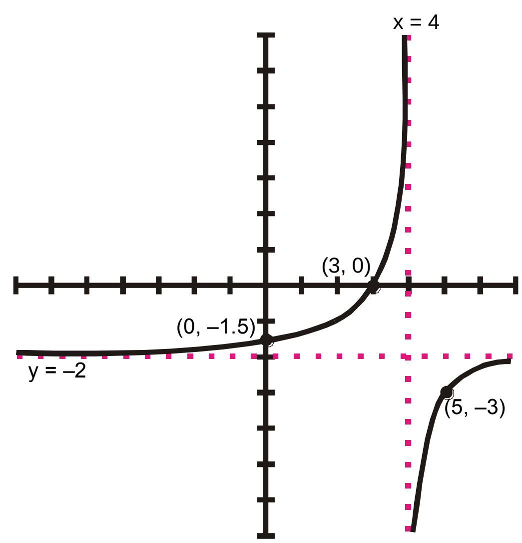
a. What is the domain of $f(x)$?
b. What is the range of $f(x)$?
c. What is $f(0)$?
d. What values of x give $f(x) = 0$?
e. What is $f(-1)$?
f. Where is $f(x)$ increasing?
g. Where is $f(x)$ positive?
h. As $x \to -\infty$, what does f do?
i. As $x \to \infty$, what does f do?



y = f(x)

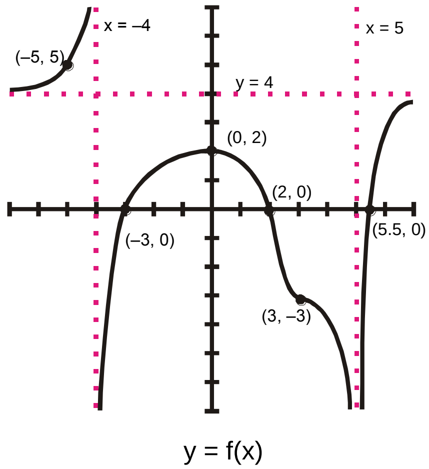
4. Given the graph of $y = f(x)$ at the right,

a. What is the domain of $f(x)$?
b. What is the range of $f(x)$?
c. What is $f(0)$?
d. What values of x give $f(x) = 0$?
e. What is $f(5)$?
f. What values of x give $f(x) = -2$
g. Where is $f(x)$ increasing?
h. Where is $f(x)$ positive?
i. As $x \to \pm\infty$, what does f do?
j. As $x \to 4$, what does f do?



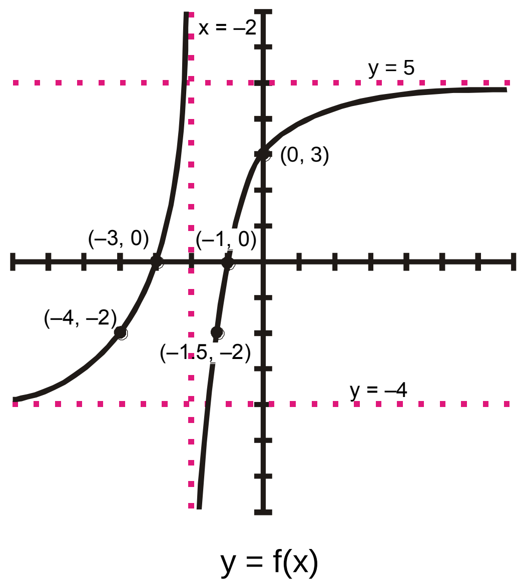
y = f(x)

5. Given the graph of $y = f(x)$ at the right,



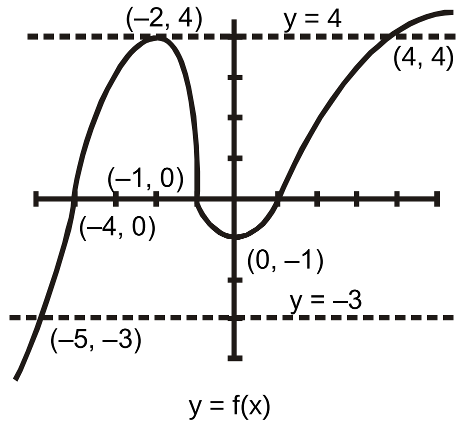
a. What is the domain of $f(x)$?
b. What is the range of $f(x)$?
c. What is $f(0)$?
d. What values of x give $f(x) = 0$?
e. What is $f(3)$?
f. What values of x give $f(x) = 5$?
g. What is $f(-4)$?
h. Where is $f(x)$ positive?
i. For $-3 < x < 2$, what is the maximum value of $f(x)$?
j. As $x \to \pm\infty$, what does f do?
k. As $x \to 5$, what does f do?
l. As $x \to -4$, what does f do?

6. Given the graph of $y = f(x)$ at the right,



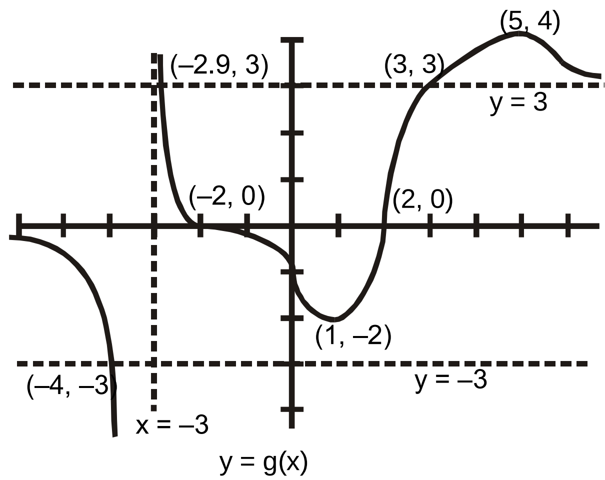
a. What is the domain of $f(x)$?
b. What is the range of $f(x)$?
c. What is $f(0)$?
d. What values of x give $f(x) = 0$?
e. What is $f(-3)$?
f. What values of x give $f(x) = -2$?
g. Where is $f(x)$ increasing?
h. Where is $f(x)$ positive?
i. As $x \to \pm\infty$, what does f do?
j. As $x \to -2$, what does f do?

7. Given the following graph of $f(x)$,



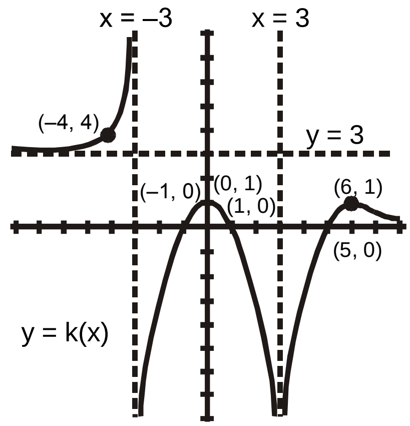
a. What is the domain of f?
b. What is the range of f?
c. What is $f(0)$?
d. What values of x give $f(x) = 0$?
e. What values of x give $f(x) \geq 4$?
f. What is $f(-5)$?
g. What domain values make $f > 0$?
h. What domain values make $f < 0$?
i. What values of x give $f(x) = 4$?
j. As $x \to \infty$, what does $f(x) \to$?
k. As $x \to -\infty$, what does $f(x) \to$?

8. Given the following graph of $g(x)$,



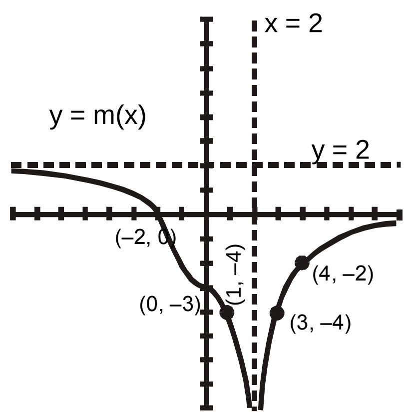
a. What is the domain of g?
b. What is the range of g?
c. What is $g(0)$?
d. For what values of x does $g(x) = 0$?
e. For what values of x does $g(x) = 3$?
f. What is $g(3)$?
g. What values of x give $g(x) = -3$?
h. As $x \to -\infty$, what does $g \to$?
i. As $x \to +\infty$, what does $g \to$?
j. As $x \to -3$, what does $g \to$?

9. Given the following graph of $k(x)$,



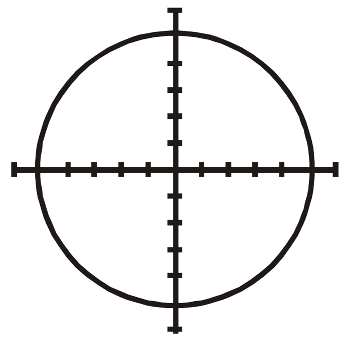
a. What is the domain of $k(x)$?
b. What is the range of $k(x)$?
c. For what values of x is $k(x) = 0$?
d. What is $k(0)$?
e. What is $k(6)$?
f. For what values of x does $k(x) = 4$
g. What is $k(3)$?
h. As $x \to -3$, what does $k \to$?
i. As $x \to 3$, what does $k \to$?
j. As $x \to \pm\infty$, what does $k \to$?

10. Given the following graph of $m(x)$,



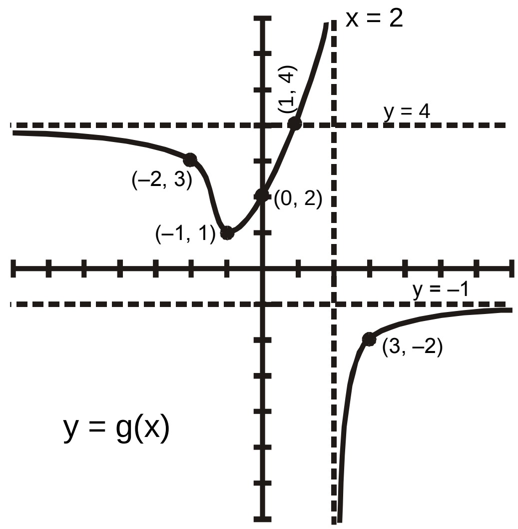
a. What is the domain?
b. What is the range?
c. What is $m(0)$?
d. For what values of x is $m(x) = 0$?
e. What is $m(4)$?
f. For what values of x does $m(x) = -4$?
g. For what values of x is $m(x)$ decreasing?
h. For what values of x is $m(x)$ increasing?
i. For what values of x is $m(x) \geq 0$?
j. As $x \to 2$, what does $m(x) \to$?
k. As $x \to \pm\infty$, what does $m(x) \to$?

11. Given the following graph of $h(x)$



a. What is the domain of h?
b. What is the range of h?
c. What is $h(0)$?
d. For what values of x does $h(x) = 0$?
e. For what values of x is $h(x) = 4$?
f. What is $h(4)$?
g. For what values of x is $h(x) \leq -4$?
h. Is h a function?

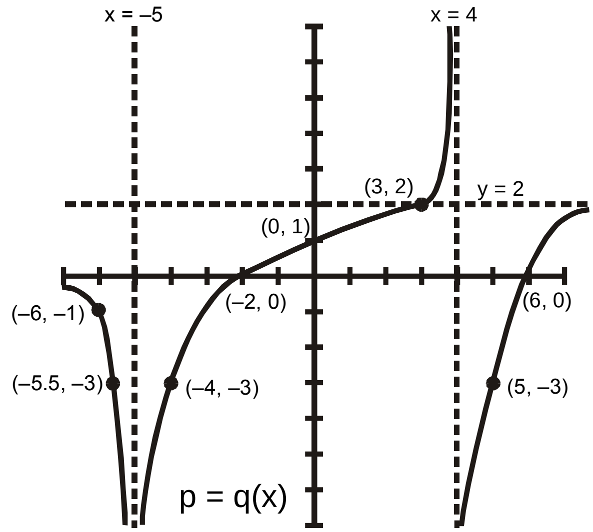
12. Given the graph $y = g(x)$



a. What is the domain of g?
b. What is the range of g?
c. What is $g(0)$?
d. What is $g(-1)$?
e. What values of x give $g(x) = 4$?
f. What values of x give $g(x) = 0$?
g. For $x < 2$, what is the minimum value of g?
h. Where is g increasing?
i. Where is g negative?
j. As $x \to \pm\infty$, $g \to$?
k. As $x \to 2$, $g \to$?

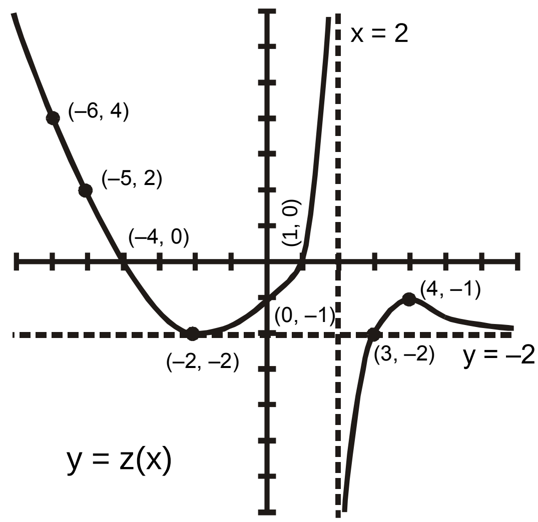
13. Given the graph of $p = q(x)$,

a. What is the domain of q?
b. What is the range of q?
c. What is $q(-5)$?
d. What is $q(0)$?
e. What values of x give $q(x) = -3$?
f. What values of x give $q(x) = 2$?
g. Where is q positive?
h. Where is q decreasing?
i. As $x \to -5, q \to$?
j. As $x \to 4, q \to$?
k. As $x \to \pm\infty, q \to$?



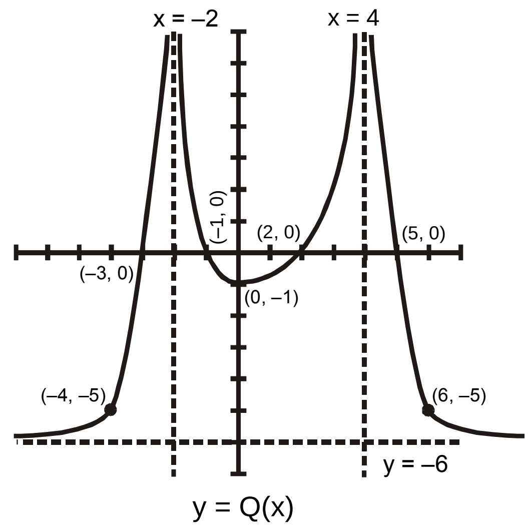
14. Given the graph of $y = z(x)$,

a. What is the domain of z?
b. What is the range of z?
c. What is $z(0)$?
d. What is $z(-5)$?
e. What values of x give $z(x) = 0$?
f. What values of x give $z(x) = -2$?
g. For $x < 2$ what is the minimum value of z?
h. For $x > 2$, what is the maximum value of z?
i. Where is z positive?
j. Where is z decreasing?
k. As $x \to \pm\infty$, $z \to$?
l. As $x \to 2$, $z \to$?



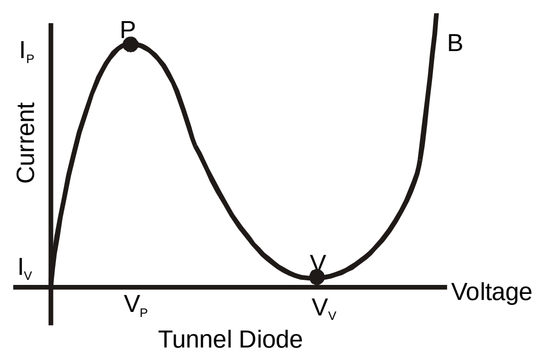
15. Given the graph of $Q(x)$,

a. What is the domain of Q?
b. What is the range of Q?
c. What is $Q(0)$?
d. What is $Q(-2)$?
e. What values of x give $Q(x) = 0$?
f. What values of x give $Q(x) = -5$?
g. What values of x give $Q(x) = -6$?
h. For $-2 < x < 2$, what is the minimum value of Q?
i. Where is Q negative?
j. Where is Q increasing?
k. As $x \to 4$, $Q \to$?
l. As $x \to \pm\infty$, $Q \to$?



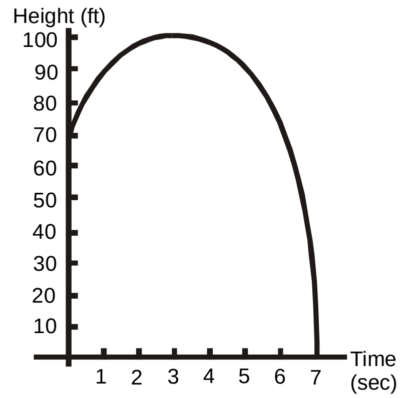
16. The graph below is the graph of the current-versus-voltage characteristics of the tunnel diode. V_p, I_p, V_v, and I_v refer to peak voltage, peak current, valley voltage, and valley current respectively. Restrict the domain of the function to $0 \leq x \leq a$ and restrict the range to $0 \leq y \leq b$.

a. What are the coordinates of the point P?
b. What are the coordinates of the point V?
c. Over what intervals is the current rising?
d. Over what intervals is the current decreasing?



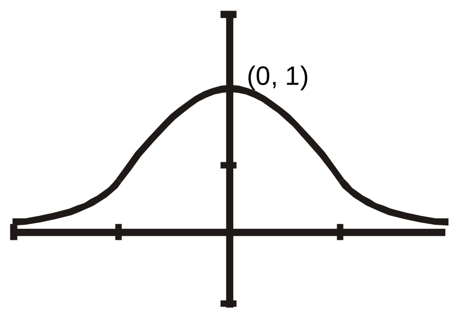
17. An object thrown into the air has a trajectory shaped like a parabola. Given the flight of an object whose trajectory is given by the function $y = s(t)$ graphed below,

a. What is the domain of $s(t)$?
b. What is the range of $s(t)$?
c. What is the maximum height reached by the object?
d. At what time does the object reach the ground?
e. From what height is the object thrown?



18. The distribution of errors in a measurement is given by the function $Erf(x)$. The graph of the function $y = E(x)$ given below resembles that of the error function.

a. What is the domain of E?
b. What is the range of E?
c. As $x \to +\infty$, what does $E \to$?
d. Over what intervals is $E > 0$?
e. Over what intervals is $E < 0$?



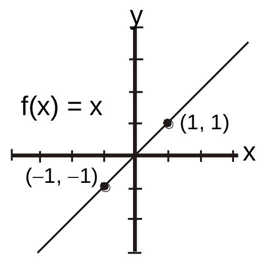
y = E(x)

2.6 Transformations of Graphs

Like many things in life, graphs come in groups or categories. The process of graphing a function can be simplified by knowing the basic structure of a particular graph type and how it relates to the specific function. For example, all quadratic equations have the same bowed shape. The only difference between the graphs of two different quadratic equations is the eccentricity (degree) of the bow and the placement or location of the graph on the coordinate system.

Because all graphs within a particular category are fundamentally the same, we can change one graph into another. This process is called a transformation.

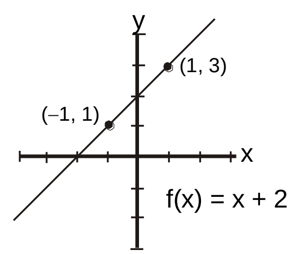
The transformation of a graph is a relocation of the points of the graph. There are two ways that we can move a point. We can actually relocate the point or we can change the axis system that we are measuring the point against. For purposes of visual simplicity we shall talk and act as if we are always moving the points of the graph. Under a given transformation the point (x, y) will be transformed into a new point (x', y').



Graph 1

As a general rule, we can break transformations into two types. Those which change the x coordinates of the graph and those which change the y coordinates. The application of a transformation to the x variable will affect the graph in a horizontal direction only. The application of a transformation to the y variable will affect the graph in the vertical direction only.

When dealing with function notation $y = f(x)$, transformations on the inside of the function symbol $f()$ are applied to the x variable and transformations applied outside the function symbol are applied to the y variable.

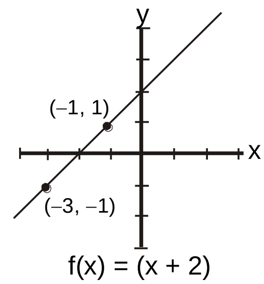


Graph 2

Consider the function $f(x) = x$ shown in graph 1. We want to see what happens to the graph when we add or multiply values to the inside of the function (directly on the x variable) or outside the function (applied to the function rule itself). In order to see how the graph changes we will look at the points (0, 0), (1, 1) and (–1, –1) on the graph of $f(x)$.

Now look at the graph of the function $y = f(x) + 2$ (Graph 2). Because $f(x) = x$ this is equivalent to $y = x + 2$. The graph of this equation is shown in graph 2. This transformation of $f(x)$ added 2 units to the outside of the function which is a change in the y direction. Looking at the graph we see that the point (0, 0) has changed to the point (0, 2) or that 2 units have been added to the y coordinate. Similarly, the point (1, 1) has become (1, 3) and the point (–1, –1) has become (–1, 1). Visually, the graph is raised 2 units.

What happens when we add 2 units inside the function (to the x variable) to get $y = f(x + 2)$? Again this is equivalent to the equation$x = x + 2$. Only this time we are changing the graph in the x direction. Now the point (0, 0) becomes (–2, 0). This is the same as subtracting 2 units from the x coordinate. Similarly, subtracting 2 from the x coordinates of (1, 1) and (–1, –1) gives us (–1, 1) and (–3, –1) respectively. Visually, the graph shifts 2 units in the negative direction (see graph 3).

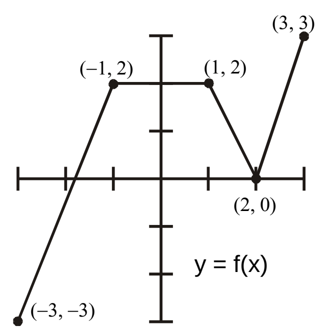


Graph 3

The student should show that subtracting 2 units outside the function lowers the graph 2 units or subtracts 2 units from each of the y coordinates. Also, subtracting 2 units inside the function shifts the graph 2 units in the positive x direction or adds 2 units to each x coordinate.

Example 1: Consider the graph of f(x). Graph the function y = f(x – 3) + 1.

Under the transformation the basic shape of the graph will remain the same. All we have to do is move a few points around to determine the new placement.



Example 1a

The transformation adds a 1 to the outside of the function. This means that we add 1 unit to each of the y coordinates of the points on the graph. The transformation also subtracts a 3 on the inside of the function. Here we add 3 units to each of the x coordinates of the graph.

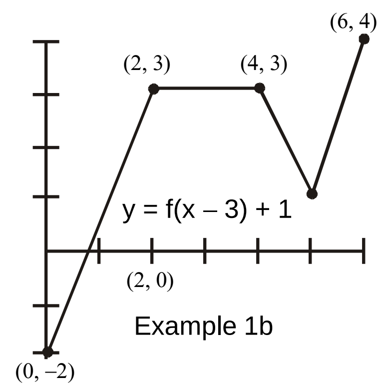
The point (–3, –3) becomes (0, –2),

(–1, 2) becomes (2, 3),

(1, 2) becomes (4, 3),

(2, 0) becomes (5, 1),

and (3, 3) becomes (6, 4).

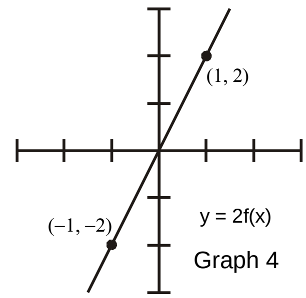


Example 1b

Plot these points and connect them similar to the original graph to get the graph of the transformed equation.

This type of transformation, which shifts the graph around the coordinate plane, is called a translation. Another type of transformation, called a scaling, distorts the graph by stretching or shrinking it with respect to the coordinate axes.

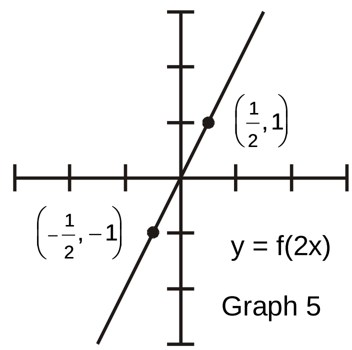
Refer again to the function $f(x) = x$. The graph of $y = 2f(x)$ is shown in graph 4. Because of the function definition, this is equivalent to $y = 2x$ which is a line with slope 2 and y intercept at the origin.



Graph 4

To see how the graph has been changed look at the points (1, 1) and (–1, –1) on the original graph. On the new graph the corresponding points are (1, 2) and (–1, –2). To convert from the first pair of points to the second we multiply the y coordinate by 2. Consequently, multiplying outside the function by 2 multiplies the y coordinates by 2. This is the same as stretching the graph by a factor of two in the y direction.

The graph of the equation $y = f(2x)$ also reduces to $y = 2x$ because of the definition of f (see graph 5). Because the transformation is inside the function symbol the transformation is only in the x direction. This time the point (1,1) becomes the point $\left(\frac{1}{2}, 1\right)$ and the point (–1, –1) becomes $\left(-\frac{1}{2}, -1\right)$. Multiplying on the inside of the function by 2 has reduced the x coordinates by a factor of 2 (or multiplied the x coordinates by $\frac{1}{2}$). This is equivalent to shrinking the graph by $\frac{1}{2}$ in the x direction.



Graph 5

Rules of Transforming and Scaling Graphs

Adding a units on the outside of the function raises the graph of the function a units (adds a units to each y coordinate of the function).

Adding a units on the inside of the function moves the graph of the function $-a$ units (subtracts a units from each of the x coordinates).

Multiplying outside the function by a units stretches the graph by a factor of a in the y direction (multiplies each y coordinate by a).

Multiplying inside the function by a units shrinks the graph by a factor of a (multiplies each x coordinate by $\frac{1}{a}$).

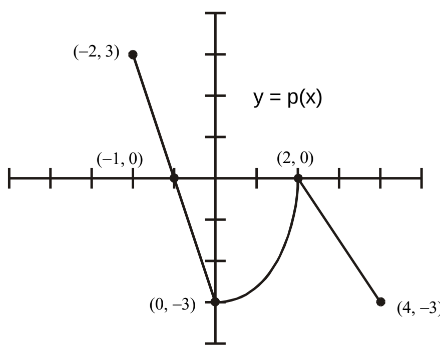
Notice that the transformation of the x coordinates is the reverse of what we might expect while the transformation of the y coordinates are direct and straightforward.

Combining translations and scalings is a relatively simple procedure as long as we follow a few simple rules.

If we wish to transform the function $f(x)$ by $y = af(cx + d) + b$ where $a, b, c,$ and d are constants, then for each point (x, y) on the original graph, the corresponding point (x', y') on the transformed graph is found by the transformation equations $y' = ay + b$ and $x' = \frac{x-d}{c}$. This latter equation is derived by letting $x = cx' + d$ and then solving for x'.

These two rules follow our above discussion of transformations. To transform the y coordinate we rescale by multiplying by a and translate by adding b. To transform the x coordinate we translate by subtracting d and then rescale by dividing by c. Note carefully the order of the operations. To transform the y coordinate we first rescale then translate. To transform the x coordinate we first translate and then we rescale. This is in accord with the 'backwards' process of transforming the x coordinate that we observed above.

Example 2: Consider the function $y = p(x)$. Draw the graph of the transformed function $y = \frac{1}{3}p\left(\frac{1}{2}x - 1\right) + 2$.



Example 2a

To transform the y coordinates of this graph we let $y' = \frac{1}{3}y + 2$. To transform the x coordinates let $x = \frac{1}{2}x' - 1$, solve for x' to get $x' = 2(x + 1)$ or $x' = 2x + 2$. We then apply these two transformations to the points (–2, 3), (–1, 0), (0, –3), (2, 0), and (4, –3) with the following results:

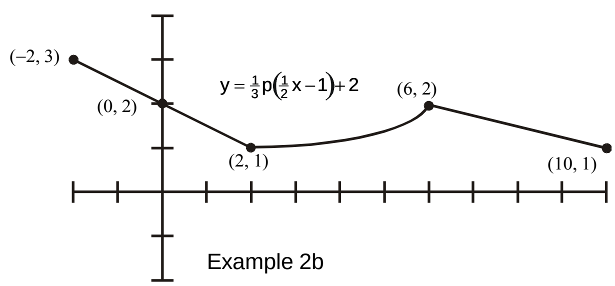
(–2, 3) → (–2, 3)

(–1, 0) → (0, 2)

(0, –3) → (2, 1)

(2, 0) → (6, 2)

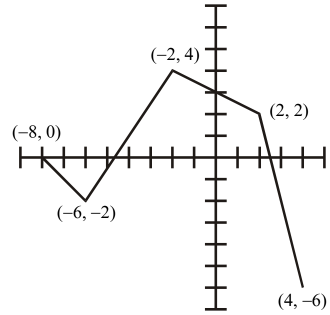
(4, –3) → (10, 1).



Example 2b

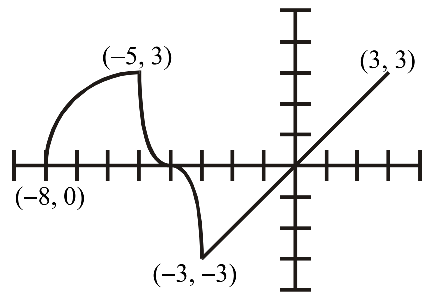
2.6 Transformations of Graphs Practice

1. Given the following graph of $y = f(x)$, graph



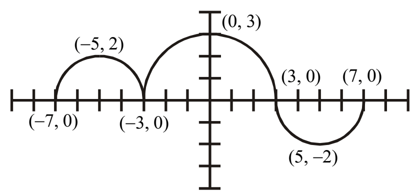
a. $y = f(x+4)$
b. $y = f(x) + 4$
c. $y = 2f(x)$
d. $y = f(2x)$
e. $y = 2f(2x+4) + 4$

2. Given the following graph of $y = g(x)$, graph



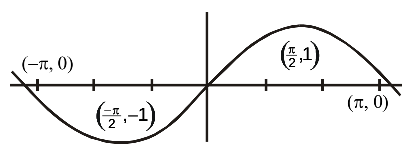
a. $y = -g(x)$
b. $y = g(-x)$
c. $y = -g(x) + 2$
d. $y = -g(x-2)$
e. $y = g(-x-2)$

3. Given the following graph of $h(x)$, graph



a. $y = \frac{1}{2}h(x)$
b. $y = h\left(\frac{1}{2}x\right)$
c. $y = \frac{1}{2}h\left(\frac{1}{2}x\right)$
d. $y = -\frac{1}{2}h(-2x)$
e. $y = 2h(-x+2) - 3$

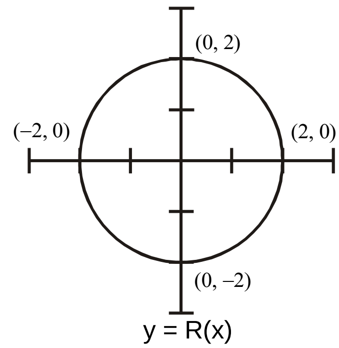
4. Given the following graph of $S(x)$, graph



a. $y = S(x) + 1$
b. $y = 2S(x)$
c. $y = S(2x - \pi)$
d. $y = \frac{1}{2}S\left(3x + \frac{\pi}{2}\right)$
e. $y = 2S\left(\frac{1}{2}x + 2\pi\right)$

5. Given the following graph of $R(x)$, graph

a. $y = R(x) + 1$
b. $y = R(x - 2)$
c. $y = R(x - 2) + 1$
d. $y = 2R(x)$
e. $y = R(3x)$
f. $y = 2R(3x - 3) + 2$



Notice that $R(x)$ is not a function. It fails the vertical line test for functions. However, our transformations still work.

6. Let the points (–2, 1), (0, 2), (1, 3), (2, 4), (4, 0) be on the function $f(x)$. Give the corresponding points for each of the following transformations.

a. $y = f(-x - 2) + 1$
b. $y = 2f(x + 3) - 2$
c. $y = \frac{3}{2}f\left(\frac{3}{2}x - 1\right) + 3$
d. $y = \frac{1}{2}f\left(-\frac{3}{2}x - 2\right) - 1$

7. Let the points (–3, –1), (–1, 2), (0, 3), (2, –2), (4, 1) be on the function $g(x)$. Give the corresponding points for each of the following transformations.

a. $y = -3g(2x - 1) + 1$
b. $y = \frac{1}{2}g(-x + 2) - 3$
c. $y = -2g\left(\frac{1}{2}x - 2\right) - 1$
d. $y = \frac{2}{3}g\left(\frac{3}{2}x + \frac{1}{2}\right) - \frac{1}{3}$

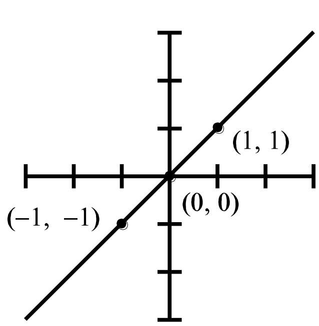
8. Let the points (–4, 3), (–3, 3), (0, –2), (2, 1), (4, –1) be on the function $h(x)$. Give the corresponding points for each of the following transformations.

a. $\mathrm{y = -2h(2x - 3) + 1}$
b. $y = \frac{1}{2}h(3x - 1) + \frac{3}{2}$
c. $y = -h(-x + 1) + 1$
d. $y = 3h\left(\frac{1}{2}x - 3\right) + 2$

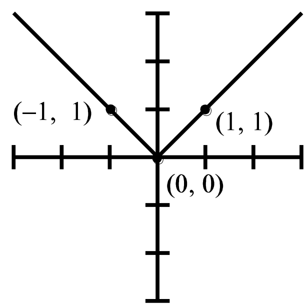
2.7 Transformations of Basic Functions

Many of the equations which you will encounter are transformations of a few basic functions. By transforming a few reference points according to the rules in the last section, we will be able to graph variations of these functions.

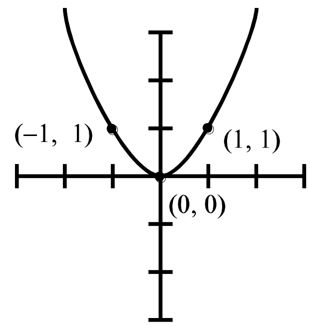
In the following we will look at six of the most common graphs. These graphs and their corresponding functions are:



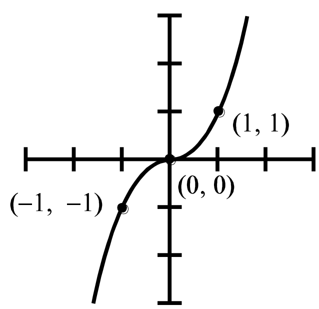
$y = x$



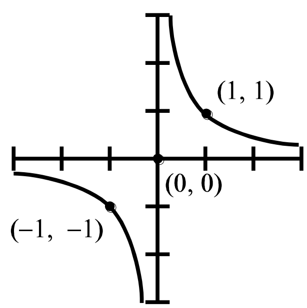
$y = |x|$



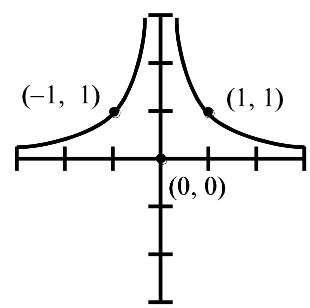
$y = x^2$



$y = x^3$



$y = \frac{1}{x}$



$y = \frac{1}{x^2}$

The Linear Function $f(x) = x$

This function is a straight line with intercepts at the origin and a slope of 1. Easy reference points on this function are (–1,–1), (0,0) and (1,1).

If we look at the slope intercept form of the equation of a line $y = mx + b$, we notice that the equation can be written $y = mf(x) + b$ where $f(x) = x$. Here m is a rescaling factor and b is a translation factor, both in the y direction.

Example 1: Graph the function $y = \frac{2}{3}x - 2$

$$y = \frac{2}{3}x - 2$$

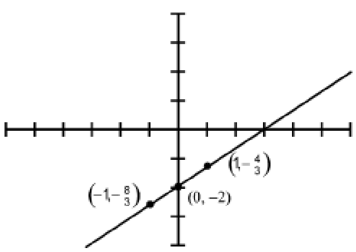
Rewrite in function notation where $f(x) = x$

$$y = \frac{2}{3}f(x) - 2$$

Multiply each of the y coordinates by $\frac{2}{3}$ and then add –2.
Leave the x coordinates alone

$$(-1,-1) \to \left(-1, -\frac{8}{3}\right)$$
$$(0,0) \to (0,-2)$$
$$(1,1) \to \left(1, -\frac{4}{3}\right)$$

Graph the points



Final answer

The Absolute Value Function $f(x) = |x|$

The three reference points that we will use on the absolute value function are (–1, 1), (0, 0), and (1, 1).

Example 2: Graph the function $y = -2\left|\frac{3}{2}x - 1\right| + 4$.

$y = -2\left|\frac{3}{2}x - 1\right| + 4$ Rewrite in function notation where $f(x) = |x|$

$y = -2f\left(\frac{3}{2}x - 1\right) + 4$ Transformations outside the function symbol affect only the y values. Therefore, we multiply each y coordinate by –2 and add 4.

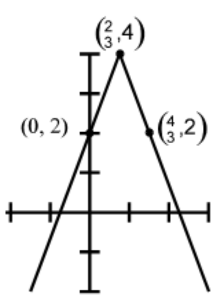
$y' = -2y + 4$ On the inside of the function symbol we change only the x values. Let $x = \frac{3}{2}x' - 1$ and solve for x'

$$x = \frac{3}{2}x' - 1$$
$$x + 1 = \frac{3}{2}x'$$
$$\frac{2}{3}(x + 1) = x'$$

Calculate new reference points

$(-1,1) \rightarrow (0,2)$
$(0,0) \rightarrow \left(\frac{2}{3}, 4\right)$
$(1,1) \rightarrow \left(\frac{4}{3}, 2\right)$

Graph the points



Final answer

The Quadratic Function $f(x) = x^2$

The quadratic will occur again in future sections, under polynomials. Here we will only concern ourselves with transformations of the graph.

The three reference points are (–1, 1), (0, 0), and (1, 1). To graph any quadratic we need only look at transformations of these points on the graph of $y = x^2$.

Example 3: Graph the function $y = \left(\frac{1}{2}x - 2\right)^2 - 1$

$$y = \left(\frac{1}{2}x - 2\right)^2 - 1$$

Rewrite in function notation where $f(x) = x^2$

$$y = f\left(\frac{1}{2}x - 2\right) - 1$$

Transformations outside the function affect only the y values. Therefore, we subtract 1.

$$y' = y - 1$$

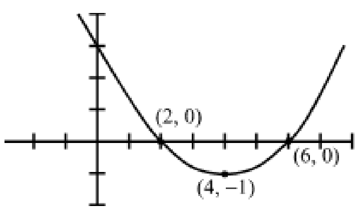
On the inside of the function symbol we change only the x values. Let $x = \frac{1}{2}x' - 2$ and sove for x'

$$x = \frac{1}{2}x' - 2$$
$$x + 2 = \frac{1}{2}x'$$
$$2(x + 2) = x'$$
$$2x + 4 = x'$$

Calculate new reference points

$$(-1,1) \to (2,0)$$
$$(0,0) \to (4,-1)$$
$$(1,1) \to (6,0)$$

Graph the points

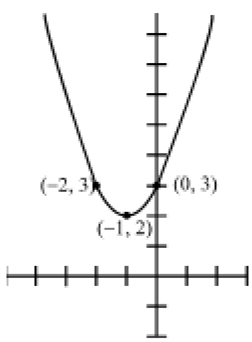


Final answer

A question that frequently arises is "What happens to the exponent 2?" The function rule $f(x) = x^2$ means "square whatever is inside the function symbol $f(\,)$". The exponent is not lost, it is replaced by another way of saying "square."

Example 4: Graph the function $y = x^2 + 2x + 3$

Before we can graph this equation we must first write it in a form showing the basic function. To do this we complete the square on the quadratic and use the function $f(x) = x^2$.

$y = x^2 + 2x + 3$	Group $x^2 + 2x$
$y = (x^2 + 2x) + 3$	Take half the middle term and square it
$\left(\frac{1}{2} \cdot 2\right)^2 = 1^2 = 1$	Add and subtract to balance
$y = (x^2 + 2x + 1) + 3 - 1$	Factor trinomial perfect square
$y = (x+1)^2 + 2$	Rewrite in function notation where $f(x) = x^2$
$y = f(x+1) + 2$	Find y' and x' (solve for x')
$y' = y + 2$ $x = x' + 1$ $x - 1 = x'$	Calculate new reference points
$(-1,1) \to (-2,3)$ $(0,0) \to (-1,2)$ $(1,1) \to (0,3)$	Graph the points
	Final answer

The Cubic Function $f(x) = x^3$

The graph of the cubic has the three reference points (–1, –1), (0, 0), and (1, 1). When graphing a cubic we apply the appropriate transformations to these three points.

Example 5: Graph the function $y = \frac{2}{5}(x-2)^3 - 2$

$$y = \frac{2}{5}(x-2)^3 - 2$$

Rewrite in function notation, where $f(x) = x^3$

$$y = \frac{2}{5}f(x-2)^3 - 2$$

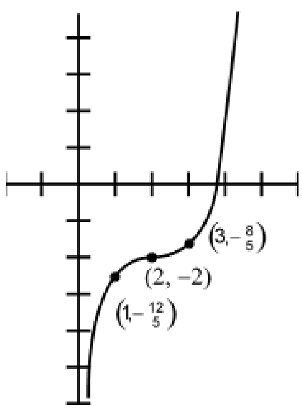
Find y' and x' (solve for x')

$$y' = \frac{2}{5}y - 2$$
$$x = x' - 2$$
$$x + 2 = x'$$

Calculate new reference points

$$(-1,-1) \rightarrow \left(1, -\frac{12}{5}\right)$$
$$(0,0) \rightarrow (2,-2)$$
$$(1,1) \rightarrow \left(3, -\frac{8}{5}\right)$$

Graph the points

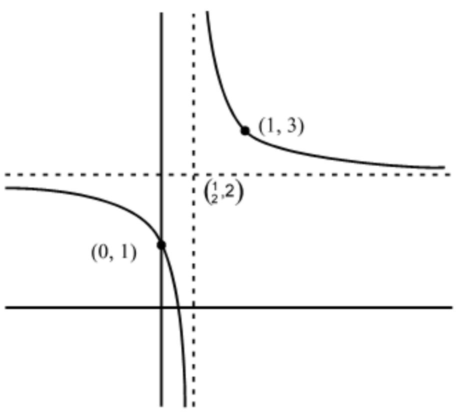


Final answer

The Reciprocal Function $f(x) = \frac{1}{x}$

The reciprocal function has both a vertical and a horizontal asymptote (which are the lines $x = 0$ and $y = 0$). The intersection of the asymptotes is the coordinate (0, 0) which is one of the three reference points for the function. The other two reference points are (–1, –1) and (1, 1). When transforming this function the transformation of the point (0, 0) becomes the intersection point of the new asymptotes. The asymptotes will retain their vertical and horizontal directions.

Example 6: Graph the function $y = \frac{1}{2x-1} + 2$

$y = \dfrac{1}{2x-1} + 2$	Rewrite in function notation where $f(x) = \frac{1}{x}$
$y = f(2x-1) + 2$	Find y' and x' (solve for x')
$y' = y + 2$ $x = 2x' - 1$ $x + 1 = 2x'$ $\dfrac{1}{2}(x+1) = x'$	Calculate new reference points
$(-1,-1) \rightarrow (0,1)$ $(0,0) \rightarrow \left(\dfrac{1}{2}, 2\right)$ $(1,1) \rightarrow (1,3)$	Graph the points
	Final answer

The vertical and horizontal asymptotes are now centered at the point $\left(\frac{1}{2}, 2\right)$. The rest of the graph is placed around these asymptotes in the same fashion as the original asymptotes.

The Inverse Square Function $f(x) = \frac{1}{x^2}$

The reciprocal function of x^2 also has horizontal and vertical asymptotes. Unlike the reciprocal function this function is always positive and its graph is above the x axis. The standard reference points are (–1, 1), (0, 0) (again the center of the asymptotes), and (1, 1).

Example 7: Graph the function $y = \frac{2}{3}\left(\frac{1}{(3x-2)^2}\right) - 4$

$$y = \frac{2}{3}\left(\frac{1}{(3x-2)^2}\right) - 4$$

Rewrite in function notation where $f(x) = \frac{1}{x^2}$

$$y = \frac{2}{3}f(3x-2) - 4$$

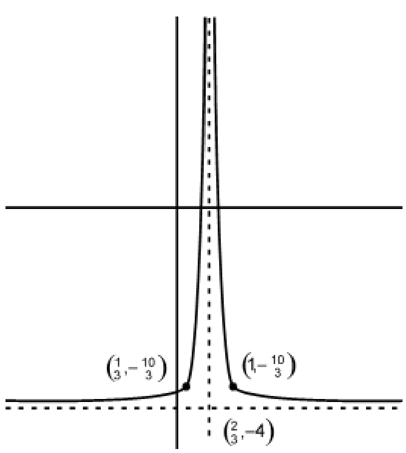
Find y' and x' (solve for x')

$$y' = \frac{2}{3}y - 4$$
$$x = 3x' - 2$$
$$x + 2 = 3x'$$
$$\frac{1}{3}(x+2) = x'$$

Calculate new reference points

$$(-1,1) \to \left(\frac{1}{3}, -\frac{10}{3}\right)$$
$$(0,0) \to \left(\frac{2}{3}, -4\right)$$
$$(1,1) \to \left(1, -\frac{10}{3}\right)$$

Graph the points



Final answer

Notice that the intersection point of the asymptotes is $\left(\frac{2}{3}, -4\right)$.

Functions Inside of Absolute Value Signs

Sometimes we need to graph an equation that has a function inside of an absolute value symbol. For example, something like $y = |x^2 - 4|$. In this example we have a quadratic function inside of the absolute value. (Please note - this is not the same as the absolute value function talked about earlier in this section although there is a relationship.)

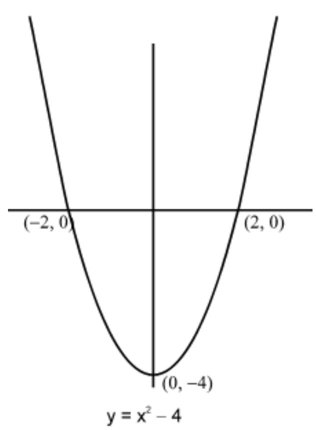
To graph this situation we need to break the graphing process into several distinct parts. First we need to graph the function inside the absolute value symbol. Then apply the absolute value operation to the graph. Lastly, apply any transformations that are outside the absolute value.

Remember from your basic algebra that absolute values make everything inside of them positive. If $y = f(x)$, then $|y| = |f(x)|$. To make the function values positive we make all the y coordinates positive. If a y coordinate is already positive we leave it alone. If negative we change the sign and replot.

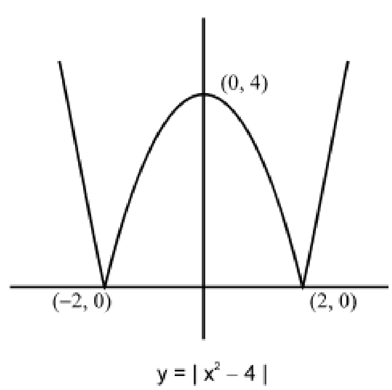
Example 8: Graph the function $y = |x^2 - 4|$

$y = |x^2 - 4|$

We start by graphing the inside equation, $x^2 - 4$



This graph is negative below the x axis, between -2 and 2
Take the portion of the graph over this interval
and fold it up over the x axis making it positive.



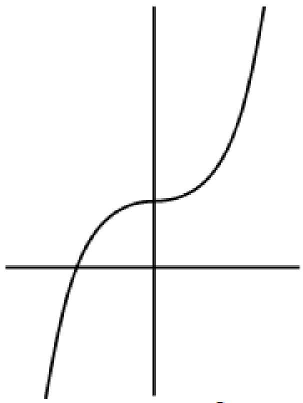
Final answer

When working with functions inside of absolute values be sure to graph the function inside the absolute values signs first. Then apply the absolute value function. If the absolute value is also to be transformed, do this only after you have done everything else.

Example 9: Graph the function $y = 2|x^3 + 1| - 1$

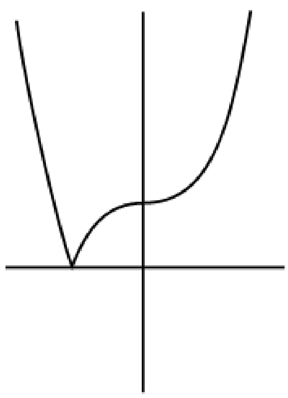
$y = 2|x^3 + 1| - 1$

First graph the equation $x^3 + 1$



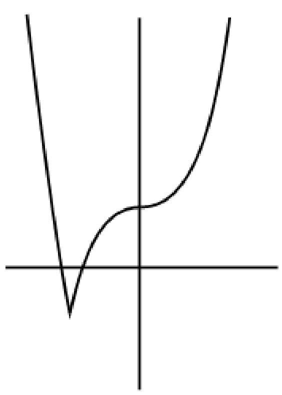
f(x) = x^3 + 1

Apply the absolute value to the graph making all points below the x axis positive (reflecting them about the x axis).



g(x) = |x^3 + 1|

The rest of the graph is a transformation of $g(x)$, $y = 2g(x) - 1$. Multiply each of the y coordinates of g by 2 and subtract 1.



y = 2|x^3 + 1| – 1

Final answer

2.7 Transformations of Basic Functions Practice

Rewrite each of the following equations in function notation using the appropriate function. Graph each equation by transforming the graph of the basic function.

1. $y = |x + 2|$

2. $y = \dfrac{1}{(x-2)^2}$

3. $y = (x - 2)^2$

4. $y = \dfrac{1}{(x+1)^2}$

5. $y = (x + 1)^3$

6. $y = 2(x + 1)^2 + 3$

7. $y = (2x - 2)^3 + 3$

8. $y = \dfrac{2}{x+1}$

9. $y = \dfrac{1}{x+1} - 1$

10. $y = 3(x + 1)^3 - 2$

11. $y = 2|x + 3| - 3$

12. $y = \dfrac{1}{2}\left(\dfrac{1}{x+2}\right) + 1$

13. $y = \dfrac{1}{2x+4} + 1$

14. $y = \dfrac{1}{2}\left(\dfrac{1}{(2x-1)^2}\right) + 1$

15. $y = \dfrac{2}{(2x-1)^2} + 2$

16. $y = x^2 + 2x + 1$

17. $y = 2x^2 + 2x + 1$

18. $y = 3x^2 + 1$

19. $y = 3\left(\dfrac{1}{2}x - 2\right)^3 + 2$

20. $y = 3\left(\dfrac{1}{2}x - 2\right)^2 - 1$

21. $y = \dfrac{3}{2x-5} + 3$

22. $y = \dfrac{1}{2(x+2)^2} - 7$

23. $y = \dfrac{1}{(3x+6)^2}$

24. $y = \dfrac{1}{2}|2x + 3| - 1$

25. $y = -2x^2 - 3x + 1$

26. $y = -2(x - 1)^3 + 1$

27. $y = \dfrac{2}{(-x+3)^2} + 1$

28. $y = \dfrac{1}{-x+1}$

29. $y = \dfrac{1}{2x^2} + 2$

30. $y = -\dfrac{1}{x+1}$

31. $y = \dfrac{1}{-x-1}$

32. $y = -|x+2|$

33. $y = 2(x+2)^2 + 2$

34. $y = \dfrac{1}{2(x+2)} + 2$

35. $y = 2(x+2)^3 + 2$

36. $y = 2x^2 - 3x + 1$

37. $y = \dfrac{x-1}{x+1}$

38. $y = \dfrac{x+1}{x-1}$

39. $y = -x^2 - x + 1$

40. $y = -x^2 + x + 1$

41. $y = x^2 - x + 1$

42. $y = -2|x-3| + 1$

43. $y = \dfrac{1}{2}(2x^2+1) - 1$

44. $y = x^3 + 3x^2 + 3x + 3$

45. $y = \dfrac{3}{2(x+2)^2} + 1$

46. $y = \dfrac{2}{3x+1} - 1$

47. $y = |x^3| + 3$

48. $y = |x^3 - 1|$

49. $y = \dfrac{1}{|x+2|}$

50. $y = \dfrac{1}{|x-2|} + 1$

51. $y = |x^2 - 2|$

52. $y = |x^2 + 2|$

53. $y = -|2x^2 + 3| - 3$

54. $y = |x^2 - 2| + 3$

55. $y = \left|\dfrac{1}{x-2}\right|$

56. $y = 2|x^2 - x - 6|$

57. $y = \big||x^2 - 4| - 4\big| - 4$

58. $y = -|x^2 - 2| + 2$

59. $\dfrac{1}{x+1} = -\dfrac{y}{2}$

60. $2(y+1) + \dfrac{1}{x} = 0$

Chapter 3:

Graphs of Key Functions

3.1 Graphs of Polynomial Functions

Polynomials are functions of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We have seen simple polynomials in the form of linear, quadratic and cubic functions. In this section we will deal with the graphs of polynomials of higher degree. For simplicity, all polynomials will have rational roots.

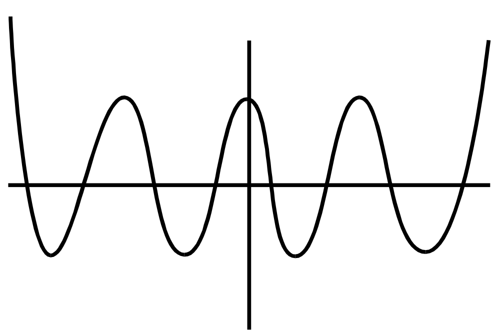
All polynomials of degree n with rational roots can be reduced and factored to the form $p(x) = (x - r_1)(x - r_2) \ldots (x - r_n)$ where $r_1, r_2, \ldots, r_n$ are the roots of the polynomial, i.e., $p(r_1) = 0,\ p(r_2) = 0$, etc. You should remember from your algebra that the root of a function is an x intercept of the graph of the function. According to the fundamental theorem of algebra every polynomial of degree n has n roots. If those roots are distinct then the graph of the polynomial has n distinct x intercepts.

Intercepts of Polynomials

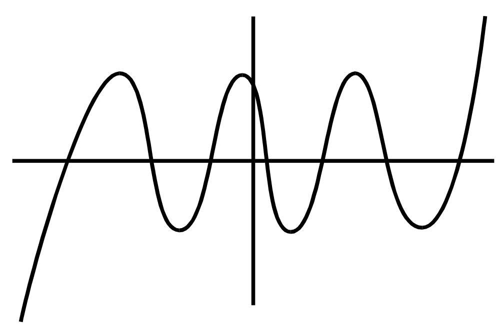
If the polynomial $P(x)$ has factors $(x - r_1), (x - r_2), \ldots, (x - r_n)$, then $P(x)$ has roots $x = r_1, x = r_2, \ldots, x = r_n$. The graph of $P(x)$ will have intercepts $(r_1, 0), (r_2, 0), \ldots, (r_n, 0)$.

Conversely, if the graph of a polynomial $P(x)$ has x intercepts $(r_1, 0), (r_2, 0), \ldots, (r_n, 0)$, then the polynomial has roots $x = r_1, x = r_2, \ldots, x = r_n$ and has factors $(x - r_1), (x - r_2), \ldots, (x - r_n)$.

The graphs of polynomials are of two kinds depending upon whether the degree of the polynomial is even or odd. Assuming that the leading coefficient of the polynomial is positive, polynomials of even degree (i.e., degree 2, 4, 6, etc.) come in from the upper left corner of the Cartesian plane, wiggle through the intercepts (roots) and head off in the upper right direction. Polynomials of odd degree (i.e., 1, 3, 5, etc.) come in from the bottom left, wiggle through the intercepts and also head off in the upper right direction. If the polynomial has a negative leading coefficient then we are multiplying outside a function by a negative which will reflect the polynomial with positive leading coefficient about the x axis (that is, turn the polynomial upside down).



EVEN LEADING EXPONENT
Enters at upper left and
leaves at upper right.



ODD LEADING EXPONENT
Enters at lower left and
leaves at upper right.

Example 1: Graph the polynomial $p(x) = (x-1)(x+1)(x+3)(x-4)$

$$p(x) = (x-1)(x+1)(x+3)(x-4)$$

The polynomial has four factors,
it is a fourth degree polynomial.
Multiply the factors we find the leading coefficient is +1.
The graph enters top left, exists top right

Set the polynomial equal to zero to find x intercepts

$$0 = (x-1)(x+1)(x+3)(x-4)$$

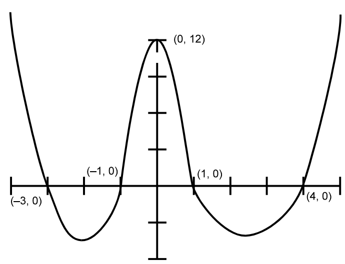
Set each factor equal to zero and solve

$$x = \pm 1, -3, 4$$

Set x equal to 0 to find the y intercept

$$p(0) = (0-1)(0+1)(0+3)(0-4)$$
$$p(0) = (-1)(1)(3)(-4)$$
$$p(0) = 12$$

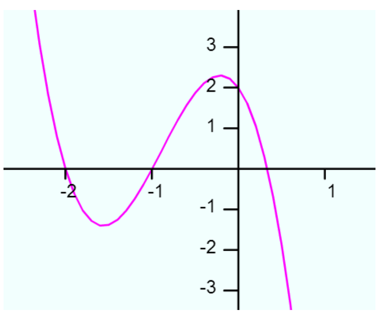
Graph x and y intercepts. Connect points to graph



p(x) = (x – 1)(x + 2)(x + 3)(x – 4)

Final answer

Example 2. Graph the polynomial $p(x) = -3x^3 - 8x^2 - 3x + 2$

$p(x) = -3x^3 - 8x^2 - 3x + 2$	The leading exponent is odd with a negative coefficient. The graph enters top left, exists bottom right
$p(x) = -(3x-1)(x+1)(x+2)$	Factor the polynomial (tricks for factoring later in this chapter)
	Set the polynomial equal to zero to find x intercepts
$0 = -(3x-1)(x+1)(x+2)$	Set each factor equal to zero and solve
$x = \frac{1}{3}, -1, -2$	Set x equal to zero (original function) to find y intercept
$p(0) = -3(0)^3 - 8(0)^2 - 3(0) + 2$ $p(0) = 2$	Graph x and y intercepts. Connect points to graph
	Final answer

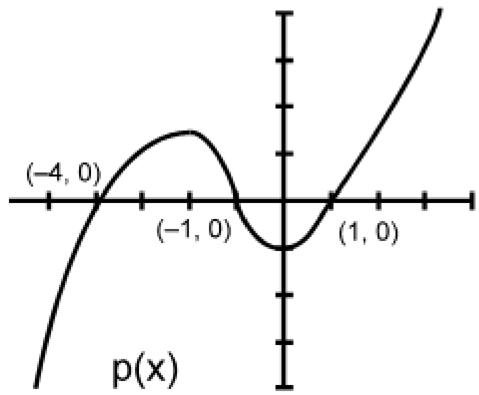
Polynomials with Multiple Roots

If a polynomial has multiple roots then we have to adjust the graph to compensate for those extra roots.

If a root has multiplicity 1 (occurs one time), then it intersects the x axis as a straight line. If a root has an even multiplicity (occurs an even number of times), then it intersects the x axis in the same way as x^2 If the root has an odd multiplicity (occurs an odd number of times greater than one), then it intersects the x axis in the same way as x^3. In each case the direction of the intersection (from above or below) is the direction which allows the curve to flow in a normal, continuous manner from left to right.

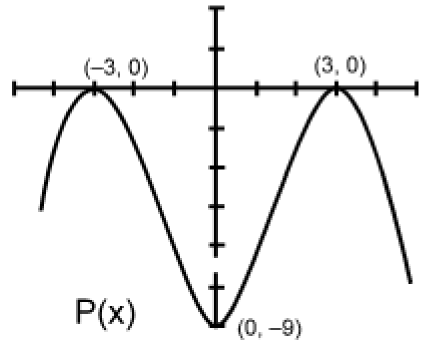
Conversely, If a curve intersects the x axis in a linear fashion (passes through the x axis without any bending), then the root occurs once. If the curve bounces off the x axis (curves like x^2 when it hits the axis), then the root has even multiplicity (occurs an even number of times). For simplicity we shall assume that the root occurs twice. If the curve flattens out a little as it passes through the x axis (looks like the graph of x^3 as it intersects the x axis), then it is a root of odd multiplicity (occurs an odd number of times). For simplicity we shall assume the root has multiplicity 3.

Example 3. Given the graph find the simplest polynomial which has the same intercepts.



This graph begins in the lower left of the plane and leaves at the upper right. This is the pattern for an odd degree polynomial with positive leading coefficient. It intercepts the x axis three times. Each intersection is linear in form (passes straight through the axis) telling us that each intercept is a root of multiplicity one. We determine the factors of the polynomial as follows: For the intercept at $x = -4$ we have the factor $(x + 4)$. For the intercept at $x = -1$ we have the factor $(x + 1)$, and for the intercept at $x = 1$ we have the factor $(x - 1)$. A polynomial with the given roots of the given multiplicities is $(x + 4)(x + 1)(x - 1) = x^3 + 4x^2 - x - 4$. The problem with this polynomial is that even though it has the correct roots (x intercepts) it has a y intercept of (0, –4) not (0, –1). To change this we need only rescale the polynomial so that it has the correct y intercept. To convert a –4 into a –1 we need to multiply by $\frac{1}{4}$. Doing this gives us the correct polynomial $p(x) = \frac{1}{4}(x^3 + 4x^2 - x - 4)$.

Example 4. Find the polynomial which best fits the following graph:



This graph starts at the lower left and leaves at the lower right. To enter and leave on the same half (bottom half of the plane in this case) the polynomial must be of even degree.

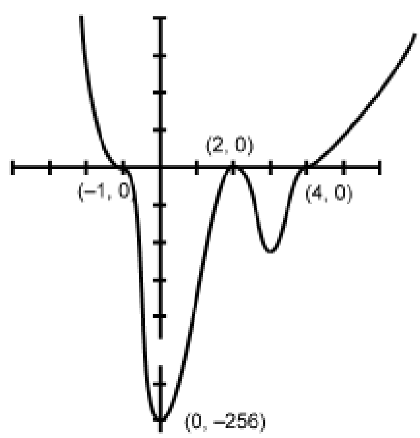
It has two intercepts $x = \pm 3$. Both of these intercepts bounce off the x axis indicating that the roots are of even multiplicity and that the factors $(x+3)$ and $(x-3)$ each occur an even number of times. For simplicity we will choose multiplicity two (actually, the roots could occur any even number of times but we have no means of deciding so we will choose the smallest even multiplicity).

Based on the above analysis we can approximate the polynomial by $p(x) = (x+3)^2(x-3)^2$. This polynomial has the appropriate roots with multiplicity two which was desired. Finding the y intercept we let $x = 0$ which gives us the point (0, 81). The y intercept on the given graph is (0, –9). To change the 81 into a –9 we rescale in the y direction by $-\frac{1}{9}$. I.e., $-\frac{1}{9}(81) = -9$. Our finished polynomial is $p(x) = -\frac{1}{9}(x+3)^2(x-3)^2$. Notice that when we rescale by $-\frac{1}{9}$ we are multiplying by a negative which reflects the graph about the x axis. This reflection turns the tails down which is what we want.

Example 5. Graph the equation $p(x) = (x+1)(x-2)^2(x-4)^3$.

This polynomial has a root of multiplicity one (–1, 0), a root of multiplicity two (2, 0), and a root of multiplicity three (4, 0). Counting up the number of roots 3 + 2 + 1 = 6, we get a polynomial of degree 6. Multiplying the leading coefficients of each factor out we get 1 which is positive. Therefore, we have a 6th degree polynomial with positive leading coefficient. The graph must enter the plane form the upper left, wiggle through the roots, and leave in the upper right.

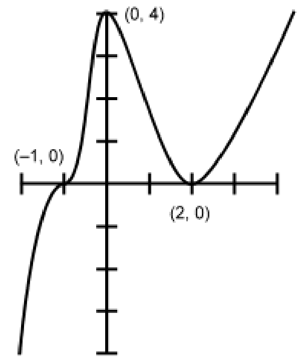
The root at (–1, 0) is a triple root and will flatten out slightly as it passes through the x axis. The root at (2, 0) is a double root and will bounce off the x axis like a quadratic. Because the curve is below the x axis as we approach this intercept the bounce will be below the axis. The root at (4, 0) is a triple root and will flatten out slightly as it passes through the x axis looking like a cubic. Letting $x = 0$ we get the y intercept (0, –256).



$p(x) = (x+1)^3(x-2)^2(x-4)^3$

Example 6. Graph the polynomial $p(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$.

This is a fifth degree polynomial with positive leading coefficient. Therefore it will start at the lower left corner of the plane, wiggle through the roots, and leave at the upper right corner of the plane.



p(x) = $x^5 - x^4 - 5x^3 + x^2 + 8x + 4$

Factoring the polynomial (using strategies discussed later in this chapter) gives $p(x) = (x + 1)^3(x - 2)^2$. The point (–1, 0) is a root of multiplicity three and the point (2, 0) is a root of multiplicity two. Letting $x = 0$ in the original equation gives the y intercept (0, 4).

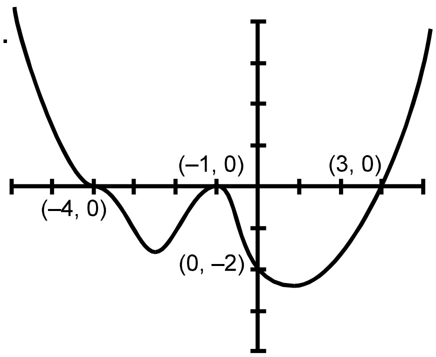
3.1 Graphs of Polynomial Functions Practice

Graph each of the following polynomial functions.

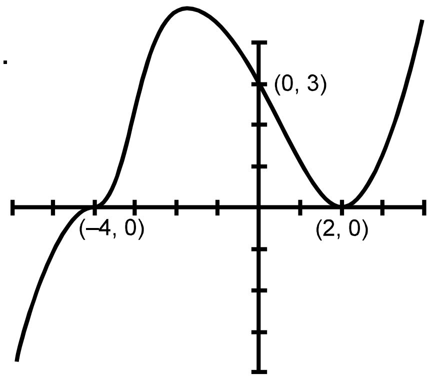
1. $y = -3x^2 + 12$
2. $y = (x+2)(x-3)$
3. $y = (x-1)(x+2)$
4. $y = (x-2)(x+2)(x+4)$
5. $y = (x+3)(x-3)(x-6)$
6. $y = x(x+4)(x+3)(x-2)$
7. $y = x(x-4)(x-2)(x-1)$
8. $y = x(2x+1)(x-1)$
9. $y = -(x+2)$
10. $y = -x(x-3)$
11. $y = 2x^2 + 5x - 3$
12. $y = 2x^2 + 9x - 5$
13. $y = x^3 - x^2 - 12x$
14. $y = 3x^3 + 7x^2 + 12x$
15. $y = -x^2 + 3x + 10$
16. $y = -x^2 + 5x - 6$
17. $y = -3x^3 - 33x^2 - 84x$
18. $y = -3x^3 - 33x^2 - 90x$
19. $y = \frac{1}{2}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x$
20. $y = \frac{1}{4}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x$
21. $y = 3x^3 + 3x^2 - 12x - 12$
22. $y = 2x^3 + 2x^2 - 18x - 18$
23. $y = (x+2)^2(x-1)$
24. $y = (2x+1)(x-1)^2$
25. $y = (3x-2)(2x-1)^2(x+1)^3$
26. $y = (3x+2)^2(3x-2)^3$
27. $y = (x+1)^2(x^2-1)^2$
28. $y = (x^4-1)^2$
29. $y = (x^4-1)^3$
30. $y = (x^4-1)^4$

Find a polynomial equation of least degree for each of the following graphs

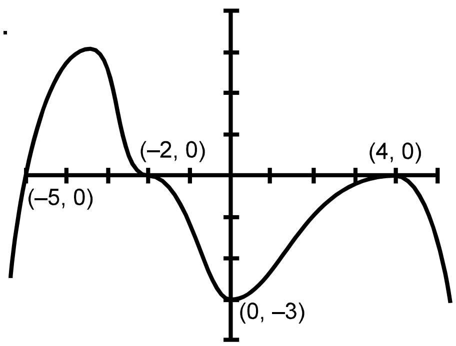
31.



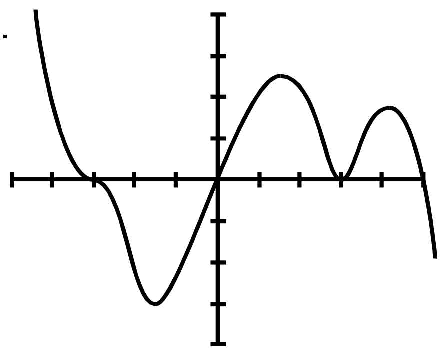
32.



33.



34.



3.2 Synthetic Division

In order to graph a polynomial it is necessary to find the roots. If we are working with quadratics (polynomials of degree 2) this is fairly straightforward. We can factor or use the quadratic formula. For polynomials of larger degree it is not always this simple. There are formulas for polynomials of degree three and four but these are not simple. It has been proven that there are no formulas to solve polynomials of degree five or larger.

As discussed in the section on graphing polynomials, finding the roots of a polynomial is equivalent to factoring the polynomial. This is relatively easy for quadratics and some simple cubic equations. Even some simple quartic equations can be easily factored. But, in general, polynomials require a little work to factor.

Factoring is really a process of division. That is, if $(x-r)$ is a factor of a polynomial $P(x)$, then the quotient $\frac{P(x)}{x-a}$ will divide with a remainder of zero. Consequently, to better understand the process of factoring a polynomial we need to understand polynomial division better.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ and $Q(x) = b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0$ be two polynomials. Consider the quotient $\frac{P(x)}{Q(x)}$ where the degree of Q is less than the degree of P, or $m < n$.
$\frac{P(x)}{Q(x)} = F(x) + \frac{R(x)}{Q(x)}$ where $F(x)$ is a polynomial of degree $n-m$ and $R(x)$ is the remainder with degree less than m. For example, let $P(x) = x^5 - 3x^3 + 2x^2 - 5$ and $Q(x) + x^3 + 3x^2 - 2x + 1$.

Then, dividing P by Q we get $\frac{P(x)}{Q(x)} = x^2 - 3x + 8 - \frac{29x^2 - 19x + 13}{x^3 + 3x^2 - 2x + 1}$ or $Q(x)$ divides $P(x)$ $x^2 - 3x + 8$ times with remainder of $-29x^2 - 19x + 13$. Or, letting $F(x) = x^2 - 3x + 8$ and $R(x) = -29x^2 - 19x + 13$ we get $\frac{P(x)}{Q(x)} = F(x) + \frac{R(x)}{Q(x)}$. If $Q(x)$ divides $P(x)$ exactly, without a remainder, then $R(x) = 0$ and $Q(x)$ is a factor of $P(x)$. I.e., $F(x)Q(x) = P(x)$.

If we take the expression $\frac{P(x)}{Q(x)} = F(x) + \frac{R(x)}{Q(x)}$ and multiply through by $Q(x)$ (eliminate the denominator), we get $P(x) = F(x)Q(x) + R(x)$. We are particularly interested when $Q(x) = x - a$. This gives us $P(x) = F(x)(x-a) + R(x)$. What happens if we evaluate $P(a)$? $P(a) = F(a)(a-a) + R(a) = R(a)$. But $Q(x)$ is linear so the remainder, $R(x)$ has to be one degree less or a constant. This means that $P(a)$ equals the remainder when we divide the polynomial by $x - a$. To show this, consider the polynomial $P(x) = x^2 - x + 2$. We want to evaluate this polynomial at $x = 2$, or find $P(2)$. $P(2) = 2^2 - 2 + 2 = 4$. Divide $P(x)$ by $x - 2$ and get $\frac{x^2 - x + 2}{x-2} = x + 1 + \frac{4}{x-2}$. Here the remainder is $4 = P(2)$. This implies that, to evaluate a polynomial $P(x)$ at $x = a$ we need only divide $P(x)$ by $x - a$ and look at the remainder.

But, as we know from our experience in beginning algebra, division of polynomials can be a long and laborious process. Therefore, we need a simpler method of dividing by the monomial $x - a$. This process is called synthetic division.

We will start with a simple division, $\frac{3x^2-5x+2}{x-3}$. As a division problem this sets up as the image on the left below.

$$
\begin{array}{r|l}
 & \mathbf{3}x + \mathbf{4} + R(\mathbf{14}) \\
x - 3 & 3x^2 - 5x + 2 \\
 & \underline{\mathbf{3}x^2 - 9x} \\
 & \quad 4x + 2 \\
 & \quad \underline{\mathbf{4}x - 12} \\
 & \qquad \mathbf{14}
\end{array}
\qquad\qquad
\begin{array}{r|lll}
 & \mathbf{3} & \mathbf{4} & \mathbf{14} \\
1 - 3 & 3 & -5 & 2 \\
 & \underline{\mathbf{3}} & \underline{-9} & \\
 & & 4 & 2 \\
 & & \underline{\mathbf{4}} & \underline{-12} \\
 & & & \mathbf{14}
\end{array}
$$

The image on the right is a simple schematic showing only the numbers. The numbers of importance to us are the bold numbers on each side.

Using the schematic on the right we can condense the picture to look like:

$$
\begin{array}{r|rrr}
-3 & 3 & -5 & 2 \\
 & & -9 & -12 \\
\hline
 & 3 & 4 & 14
\end{array}
$$

with the top row on the bottom.

Because each step of the division requires a subtraction we can change the sign on the divisor (–3) and on the second row (–9 –12) and rewrite to get:

$$
\begin{array}{r|rrr}
3 & 3 & -5 & 2 \\
 & & 9 & 12 \\
\hline
 & 3 & 4 & 14
\end{array}
$$

This has the advantage of changing our steps to addition rather than subtraction. Also, you can think of the divisor $x - 3$ as $x - 3 = 0$ or $x = 3$. So instead of dividing by $x - 3$, we are finding the value of $P(3)$. In this case $P(3) = 14$, the remainder.

This structure gives us a very simple way to divide any polynomial by the factor $x - a$ or to determine the value of $P(a)$. And, if $P(a) = 0$, or the remainder is zero, then $x - a$ is a factor of the polynomial or $x = a$ is a root. The actual procedure of the division is shown below.

step 1 $\quad \underline{3}| \; 3 \quad -5 \quad 2$ Drop the leading term

$$\begin{array}{r|rrr} 3 & 3 & -5 & 2 \\ & & & \\ \hline & 3 & & \end{array}$$

Step 2 $\quad \underline{3}| \; 3 \quad -5 \quad 2$ Multiply divisor × 3 and place in second position

$$\begin{array}{r|rrr} 3 & 3 & -5 & 2 \\ & & 9 & \\ \hline & 3 & & \end{array}$$

Step 3 $\quad \underline{3}| \; 3 \quad -5 \quad 2$ Add

$$\begin{array}{r|rrr} 3 & 3 & -5 & 2 \\ & & 9 & \\ \hline & 3 & 4 & \end{array}$$

Step 4 $\quad \underline{3}| \; 3 \quad -5 \quad 2$ Multiply and place in next position

$$\begin{array}{r|rrr} 3 & 3 & -5 & 2 \\ & & 9 & 12 \\ \hline & 3 & 4 & \end{array}$$

Step 5 $\quad \underline{3}| \; 3 \quad -5 \quad 2$ Add

$$\begin{array}{r|rrr} 3 & 3 & -5 & 2 \\ & & 9 & 12 \\ \hline & 3 & 4 & 14 \end{array}$$

The bottom line gives the quotient. The first number is the coefficient of the leading term (one degree less than the dividend) and the last number is the remainder. Because we started with a quadratic the quotient is linear, $3x + 4$ with a remainder of 14.

Example 1: Divide $\frac{4x^4-3x^2+2x-3}{x-1}$

The denominator $x - 1$ gives us a divisor of $x = 1$. Setting up the coefficients we get:

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & & & & \\ \hline & & & & & \end{array}$$

Notice the zero in the x^3 position. We need to account for all of the terms and the x^3 has a coefficient of zero. Bring down the first number.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & & & & \\ \hline & 4 & & & & \end{array}$$

Multiply 1×4 and place under the zero in the second position.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & & & \\ \hline & 4 & & & & \end{array}$$

Add.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & & & \\ \hline & 4 & 4 & & & \end{array}$$

Multiply 1×4 and place in the next position.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & 4 & & \\ \hline & 4 & 4 & & & \end{array}$$

Add.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & 4 & & \\ \hline & 4 & 4 & 1 & & \end{array}$$

Multiply 1×1 and place in the next position.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & 4 & 1 & \\ \hline & 4 & 4 & 1 & & \end{array}$$

Add.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & 4 & 1 & \\ \hline & 4 & 4 & 1 & 3 & \end{array}$$

Multiply 1×3 and place in the next position.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & 4 & 1 & 3 \\ \hline & 4 & 4 & 1 & 3 & \end{array}$$

Add.

$$\begin{array}{r|rrrrr} 1 & 4 & 0 & -3 & 2 & -3 \\ & & 4 & 4 & 1 & 3 \\ \hline & 4 & 4 & 1 & 3 & 0 \end{array}$$

This says that $\frac{4x^4-3x^2+2x-3}{x-1} = 4x^3 + 4x^2 + x + 3 + R(0)$. Because the remainder is zero, $x - 1$ is a factor of $4x^4 - 3x^2 + 2x - 3$ and $x = 1$ is a root of $4x^4 - 3x^2 + 2x - 3$.

Example 2: Divide $\frac{6x^3+7x^2-6x-2}{2x+1}$.

The divisor, $2x+1$ does not have a 1 coefficient on the x. But we are dividing to see if $2x+1$ is a factor or $2x+1=0$. Solving, this means that $x=-\frac{1}{2}$. So our divisor for the synthetic division is $-\frac{1}{2}$. Setting up the division we get:

$$\begin{array}{r|rrrr} -\frac{1}{2} & 6 & 7 & -6 & -2 \\ & & -3 & -2 & 4 \\ \hline & 6 & 4 & -8 & 2 \end{array}$$

Therefore, $\frac{6x^3+7x^2-6x-2}{2x+1}=6x^2+4x-8+R(2)$. This tells us that $2x+1$ is not a factor of $6x^3+7x^2-6x-2$ because it has a remainder of 2 when divided by $2x+1$. Also, if $f(x)=6x^3+7x^2-6x-2$, then $f\left(-\frac{1}{2}\right)=2$.

3.2 Synthetic Division Practice

Use synthetic division to find the following quotients.

1. $\dfrac{3x^3 - 2x^2 + 4x - 75}{x - 3}$

2. $\dfrac{3x^3 + 2x^2 - 4x + 8}{x + 2}$

3. $\dfrac{x^4 - 6x^3 + x^2 - 8}{x + 1}$

4. $\dfrac{x^4 - 3x^3 + x + 6}{x - 2}$

5. $\dfrac{2x^4 - 15x^2 + 8x - 3}{x + 3}$

6. $\dfrac{3x^4 - 15x - 2}{x - 2}$

7. $\dfrac{4x^5 - 4x^4 - 5x^3 + 4}{x - 1}$

8. $\dfrac{4x^3 - 3x^2 - 5x + 2}{x - 2}$

9. $\dfrac{3x^3 - 2x^2 + 4x - 24}{x - 2}$

10. $\dfrac{3x^3 + 2x^2 + 4x + 24}{x + 2}$

11. Use synthetic division to find $f(2)$ if $f(x) = 2x^3 - 3x^2 + 4x - 10$

12. Use synthetic division to find $f(3)$ if $f(x) = 3x^3 - 7x^2 - 5x + 2$

13. Use synthetic division to find $f(2)$ if $f(x) = x^4 - 5x^3 + 2x^2 - x + 3$

14. Use synthetic division to find $f(4)$ if $f(x) = x^4 - 3x^3 - 4x^2 + 2x - 5$

15. Use synthetic division to find $f(-2)$ if $f(x) = 2x^4 + 5x^3 + 2x^2 + 5x + 2$

16. Use synthetic division to find $f(-3)$ if $f(x) = 5x^4 + 10x^3 - 20x^2 - 12x - 2$

17. Use synthetic division to find $f(5)$ if $f(x) = x^4 - 20x^2 - 10x - 50$

18. Use synthetic division to find $f(-2)$ if $f(x) = x^5 - 3x^4 + 2x^2 - 5$

19. Find the value of k for which $x + 1$ is a factor of $4x^3 - 4x^2 + kx + 4$

20. Find the value of h for which $x - 2$ is a factor of $3x^3 - 5x^2 + hx + 4$

21. What is m in $3x^3 + mx^2 - 7x + 6$ if $x + 3$ is a factor?

22. What is n in $4x^3 - 2nx^2 - 8x + 6$ if $x - 3$ is a factor?

23. If $x - 4$ is a factor of $6x^3 + 13x^2 + 2kx - 40$, what is the value of k?

24. If $x + 1$ is a factor of $3x^4 - x^3 + hx^2 + x + 2$, what is the value of h?

25. Find m so that $x+1$ is a factor of $5x^3+m^2x^2+2mx-3$

26. Find n so that $x-4$ is a factor of $x^3-n^2x^2-8nx-16$

27. Find k and m so that $x^3-mx^2+2x-8k$ is divisible by $(x-1)(x+2)$

3.3 Rational Root Theorem

We want to consider roots of polynomials with real, rational coefficients. If the coefficients are complex or irrational then the process is more complicated than we need to consider.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$. We want to find the roots of this polynomial or the values of x that make $P(x) = 0$. This means we want to find factors of the form $(x - r_1)(x - r_2)(x - r_3)(x - r_4) \ldots (x - r_n) = 0$ which will give us roots $r_1, r_2, r_3, \ldots r_n$. The initial problem is that the product $(x - r_1)(x - r_2)(x - r_3)(x - r_4) \ldots (x - r_n)$ has a leading coefficient of 1, not a_n. But this is easily fixed. Because we are finding roots or points where P(x) = 0 we can divide both sides of the equation by a_n getting

$$x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \frac{a_{n-2}}{a_n}x^{n-2} + \cdots + \frac{a_0}{a_n} = 0$$

This means that each r_i has to be a factor of the fraction $\frac{a_0}{a_n}$ or that $r_i = \frac{p_i}{q_i}$ where the p_i is a factor of a_0 and the q_i is a factor of the a_n. This may sound complicated but an example will clarify the concept.

Example 1: Find the roots of the polynomial $P(x) = 8x^4 + 30x^3 - 33x^2 - 106x - 24$. According to the above discussion we are looking for fractions of the form $\frac{p}{q}$ where the p's are factors of 24 and the q's are factors of 8. All the factors of 8 are 1, 2, 4, and 8. All the factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. We need to look at all ratios with the factors of 24 in the numerator and factors of 8 in the denominator. These are:

$$\frac{1}{1},\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{2}{1},\frac{2}{2},\frac{2}{4},\frac{2}{8},\frac{3}{1},\frac{3}{2},\frac{3}{4},\frac{3}{8},\frac{4}{1},\frac{4}{2},\frac{4}{4},\frac{4}{8},\frac{6}{1},\frac{6}{2},\frac{6}{4},\frac{6}{8},\frac{8}{1},\frac{8}{2},\frac{8}{4},\frac{8}{8},\frac{12}{1},\frac{12}{2},\frac{12}{4},\frac{12}{8},\frac{24}{1},\frac{24}{2},\frac{24}{4},\frac{24}{8}$$

There are a lot of possibilities but reducing the fractions and throwing out the duplicates we get: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 2, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, 4, 6, 8, 12,$ and 24. This is a smaller set of numbers. But there is still one complication. We don't know if the roots are positive or negative. So we need to consider both possibilities for each value. This means, if the given polynomial has rational roots they must be in the collection

$$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{3}{8}, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

This is still a lot of possibilities and we will have to do a lot of guesswork. There is one rule that might simplify our guessing called Descartes rule of signs. This says that the number of positive roots of the polynomial does not exceed the number of sign changes. If all the roots are real numbers then the number of positive roots is equal to the number of sign changes. If there are any complex roots then the number of positive roots is less than the number of sign changes.

Looking at our original polynomial, $8x^4 + 30x^3 - 33x^2 - 106x - 24$ there is exactly one sign change between the cubic and quadratic terms. This means that we have at most one positive term. Because this is a fourth degree polynomial with four roots the remaining roots must be negative, assuming all real roots. To find roots we need to check negative values because they are more likely to work. Let's try $-\frac{1}{2}$. using synthetic division we get:

$-\frac{1}{2}$	8	30	–33	–106	–24
		–4	–13	23	
	8	26	-46	-83	

Multiplying the $-\frac{1}{2}$ times the –83 gives us a fraction that, when added to the –24, cannot give us a zero. If we don't get a zero in the last position $-\frac{1}{2}$ cannot be a root of the equation. Let's try $-\frac{1}{4}$. The synthetic division is

$-\frac{1}{4}$	8	30	–33	–106	–24
		–2	–7	10	24
	8	28	–40	–96	0

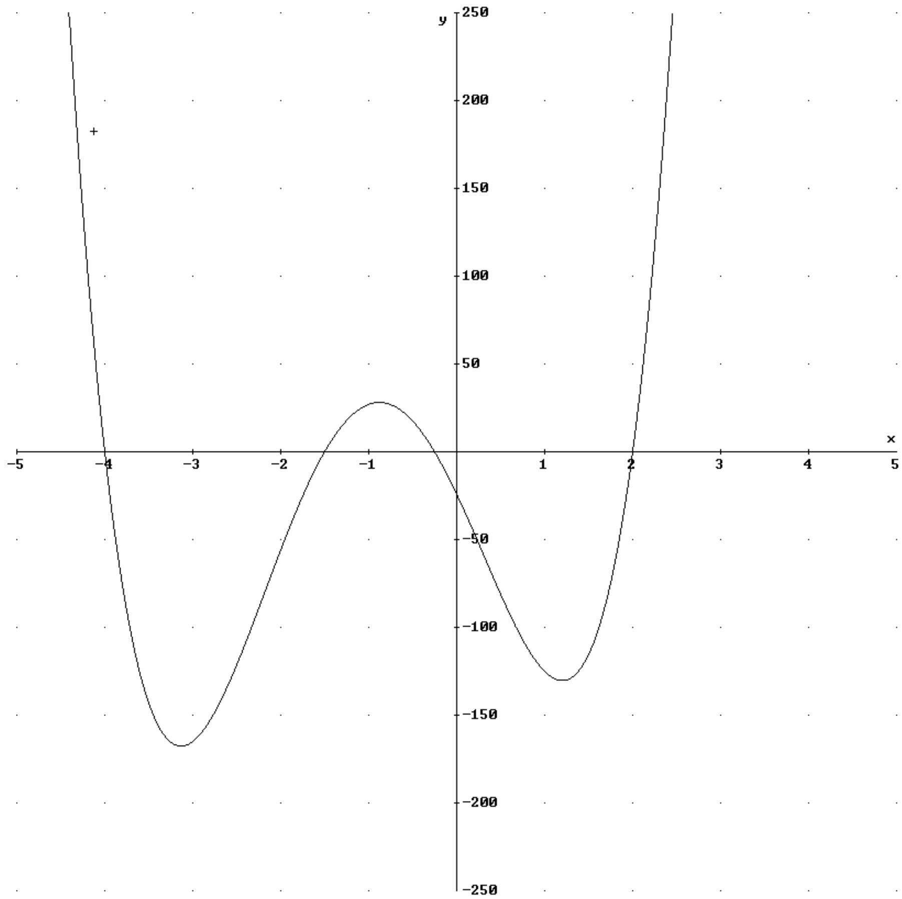
Here the remainder is zero making $x = -\frac{1}{4}$ a root of the polynomial. We have an added benefit to finding a root. Dividing out the corresponding factor from our original polynomial reduces it to $8x^3 + 28x^2 - 40x - 96$. To find zeros of this polynomial set it equal to zero and divide out the common factor of 4 to get $2x^3 + 7x^2 - 10x - 24 = 0$. Checking our factors of 24 over 2 we get a smaller set of possibilities,

$$\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

We still have only one sign change between the quadratic and linear term so still 2 negative and one positive roots. Our best guess is still to go with the negative possibilities. Let's try $-\frac{3}{2}$. Our synthetic division is:

$-\frac{3}{2}$	2	7	–10	–24
		–3	–6	24
	2	4	–16	0

The zero in the remainder position tells us that the $-\frac{3}{2}$ is a root of the polynomial. This leaves us the $2x^2 + 4x - 16$. Setting it equal to zero and factoring out the 2 gives $x^2 + 2x - 8 = 0$. This factors into $(x + 4)(x - 2) = 0$ giving our two remaining roots as $x = -4$ and $x = 2$. Therefore, our polynomial $P(x) = 8x^4 + 30x^3 - 33x^2 - 106x - 24$ has roots $-\frac{1}{4}, -\frac{3}{2}, -4$, and 2. This is an even polynomial with positive coefficient so it can be graphed:



3.3 Rational Root Theorem Practice

Use the rational root theorem and synthetic division to find all factors of the following polynomials.

1. $6x^3 - 19x^2 + x + 6$
2. $12x^3 + 20x^2 - 21x - 36$
3. $18x^3 + 9x^2 - 23x + 6$
4. $27x^3 - 69x^2 - 152x - 16$
5. $18x^3 - 27x^2 + x + 4$
6. $8x^3 - 13x^2 - 198x - 72$
7. $42x^3 + 121x^2 - 17x - 6$
8. $10x^3 + 13x^2 - 73x + 14$
9. $15x^3 + 31x^2 + x - 15$
10. $18x^3 + 3x^2 - 28x + 12$
11. $8x^3 + 12x^2 - 18x - 27$
12. $x^4 - 13x^2 + 36$
13. $x^4 - 3x^2 + 2$
14. $4x^4 - 11x^2 - 3$
15. $6x^4 - 13x^3 - 18x^2 + 7x + 6$
16. $6x^4 + 35x^3 + 75x^2 + 70x + 24$
17. $12x^4 + 20x^3 - 25x^2 - 40x - 12$
18. $4x^4 - 4x^3 - 15x^2 + 16x - 4$
19. $8x^4 - 44x^3 + 54x^2 - 25x + 4$
20. $6x^4 - x^3 - 14x^2 - x + 6$
21. $6x^4 - 7x^3 - 36x^2 + 7x + 6$
22. $4x^4 + 3x^2 - 1$
23. $24x^4 + 20x^3 - 18x^2 + 5x - 6$
24. $x^4 - x^3 + 2x^2 - 4x - 8$
25. $2x^4 + x^3 - 25x^2 - 12x + 12$
26. $4x^4 + 12x^3 + 17x^2 + 27x + 18$
27. $6x^4 + 5x^3 - 25x^2 - 10x + 24$
28. $6x^4 - 5x^3 - 25x^2 + 10x + 24$
29. $6x^4 + 7x^3 - 36x^2 - 7x + 6$
30. $12x^5 - 4x^4 - 75x^3 - 65x^2 + 3x + 9$
31. $18x^5 - 21x^4 - 139x^3 + 258x^2 - 32x - 96$
32. $32x^5 - 24x^4 - 232x^3 - 6x^2 + 182x - 60$
33. $6x^5 - 11x^4 - 52x^3 + 113x^2 + 4x - 60$
34. $24x^6 + 52x^5 - 138x^4 - 501x^3 - 523x^2 - 228x - 36$
35. $72x^6 - 324x^5 + 286x^4 + 169x^3 - 158x^2 - 27x + 18$

3.4 Graphs of Reciprocal Functions

Reciprocal functions are functions of the form $f(x) = \frac{1}{P(x)}$ where $P(x)$ is a function. The simplest reciprocal function is the function $f(x) = \frac{1}{x}$. We have seen the graph of this function before as one of the basic functions.

The Big-Little Principle

The big-little principle of reciprocal numbers says that the reciprocal of a big number is little and the reciprocal of a little number is big. For example, the reciprocal of 1,000 is .001 and the reciprocal of .0001 is 10,000. Also, the reciprocal of a positive number is positive and the reciprocal of a negative number is negative.

We can use these facts to make the graphing of reciprocal functions easier.

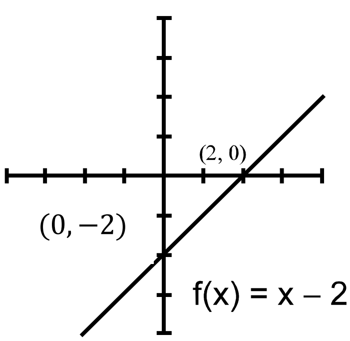
The reciprocal of a value which is far away from the x axis (large) is a value which is close to the x axis (small). The reciprocal of a value which is close to the x axis (small) is a value which is far away from the x-axis (large).

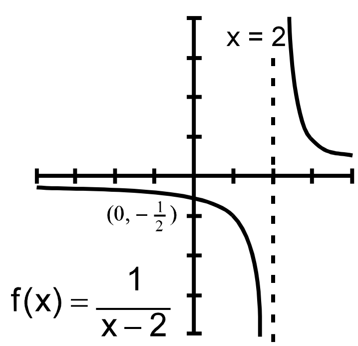
The reciprocal of the function value at a vertical asymptote ($f(x) = \pm\infty$) is 0 (an x intercept), and the reciprocal of an x intercept of a function is a vertical asymptote.

Example 1. Graph the function $f(x) = \frac{1}{x-2}$.

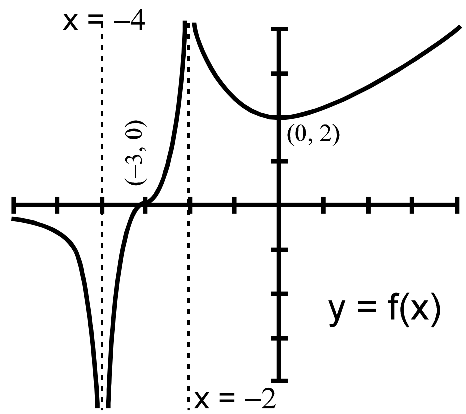
In graphing this function we could treat it as a translation of $\frac{1}{x}$, however we can also graph it as the reciprocal of $y = x - 2$. Looking at the graph of $y = x - 2$ we see that it has an x intercept at (2, 0). This tells us that the reciprocal has a vertical asymptote at $x = 2$.

Also $y \to -\infty$ as $x \to -\infty$ and $y \to +\infty$ as $x \to +\infty$. This tells us that the reciprocal function approaches 0 from below the x axis (negative side) as $x \to -\infty$ and approaches 0 from above the x axis (positive side) as $x \to +\infty$.





Example 2. Given the following graph of $f(x)$, graph $\frac{1}{f(x)}$.

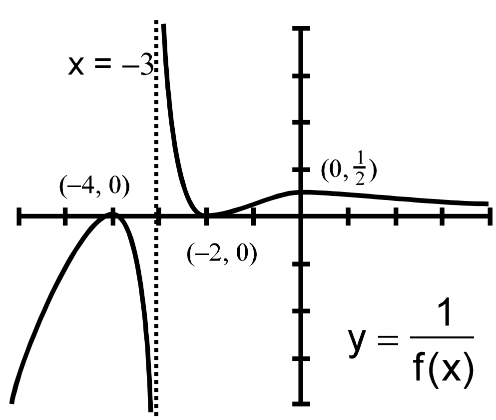


This function has two vertical asymptotes and one x intercept. It has the x axis for a horizontal asymptote and a y intercept of (0,2).

The reciprocal of this function will have a y intercept of $\left(0, \frac{1}{2}\right)$; we know this because the reciprocal of 2 is $\frac{1}{2}$. When graphing a reciprocal function we are really plotting the reciprocals of the y coordinates.

The reciprocal will have x intercepts at (–2, 0) and (–4, 0) (the values of the vertical asymptotes). It will also have a vertical asymptote at $x = -3$ (the intercept of f becomes the vertical asymptote of $\frac{1}{f}$). As $x \to +\infty, f \to +\infty$. Therefore, $\frac{1}{f}$ will approach the x axis from above (the positive side of the x-axis).

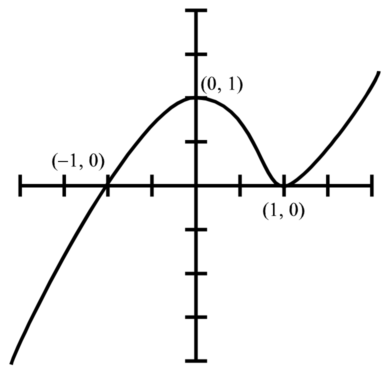
As $x \to -\infty, f \to 0$. Thus, the reciprocal $\frac{1}{f}$ will go to $-\infty$ ($f(x)$ is below the axis turning the reciprocal down towards $-\infty$).



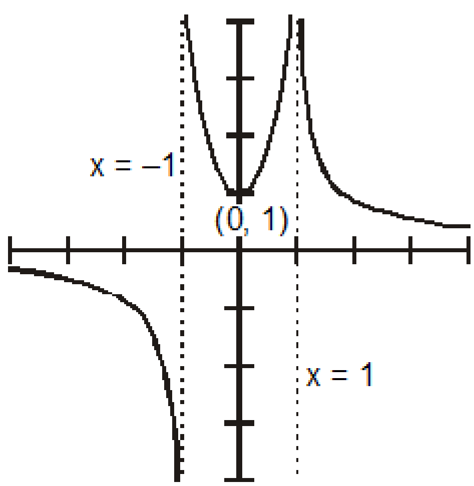
Example 3. Graph the function $f(x) = \frac{1}{x^3 - x^2 - x + 1}$.

We start by graphing $x^3 - x^2 - x + 1$ and then graphing its reciprocal.

$x^3 - x^2 - x + 1$ factors into $(x-1)^2(x+1)$. Using the techniques from the last section on graphing polynomials we get the graph below.



The graph of the reciprocal of this function will have vertical asymptotes at $x = -1$ and $x = 1$. Its y intercept will be (0, 1) (the reciprocal of 1 is 1). It will have no x intercepts but will have the x axis as a horizontal asymptote, approaching it on the negative side for $x \to -\infty$ and approaching it on the positive side for $x \to +\infty$. Notice that the graph of the reciprocal is positive on both sides of the asymptote $x = 1$. This is because $x^3 - x^2 - x + 1$ is positive on both sides of the root (0, 1).



$$f(x) = \frac{1}{x^3 - x^2 - x + 1}$$

3.4 Graphs of Reciprocal Functions Practice

Graph each of the following functions:

1. $f(x) = \dfrac{1}{x^2 - 1}$

2. $f(x) = \dfrac{1}{x^3 - 1}$

3. $f(x) = \dfrac{1}{x^2 + 1}$

4. $f(x) = \dfrac{1}{(x-2)^2}$

5. $f(x) = \dfrac{1}{x^2 - 2x - 3}$

6. $f(x) = \dfrac{2}{x(x-1)}$

7. $f(x) = \dfrac{2}{(x-1)(x-3)}$

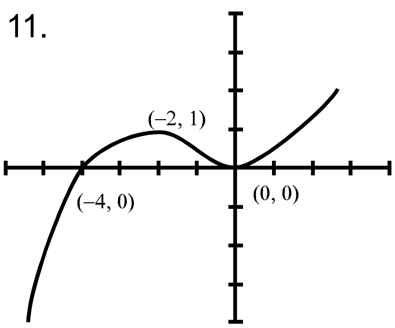
8. $f(x) = \dfrac{-1}{(x-1)(x-3)}$

9. $f(x) = \dfrac{-1}{(x^2-1)(x^2+1)}$

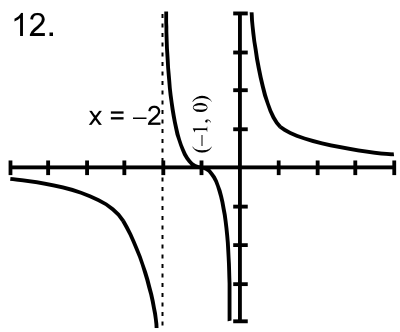
10. $f(x) = \dfrac{1}{(x+1)(x+2)(x-3)}$

Graph the reciprocal of each of the following graphs:

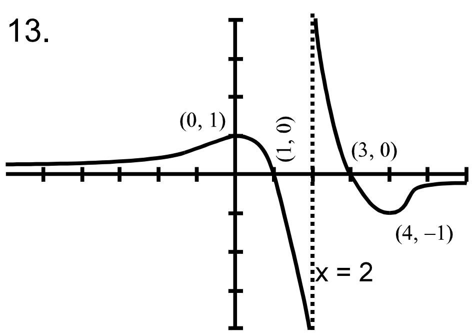
11.



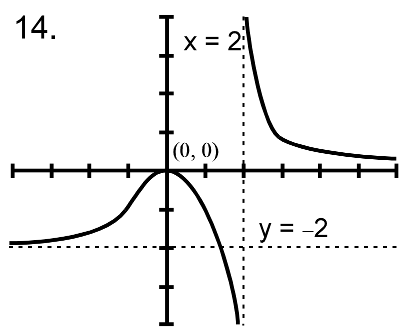
12.



13.



14.



3.5 Graphs of Rational Functions

Rational functions are functions of the type $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are both polynomial functions. The primary difference between rational functions and reciprocal functions is the polynomial in the numerator rather than a constant.

There are five particular features of a rational function that we use when graphing them. We will go through each of the five characteristics one at a time.

1. ***X* intercepts:** As in all other cases, the x intercepts are those places where the function values are 0. To find the x intercept of a rational function we set the numerator equal to 0 ($P(x) = 0$) and solve for x.

2. ***Y* intercepts:** To find the y intercept of the function we evaluate $f(0)$.

3. **Vertical asymptotes:** Like reciprocal functions rational functions also have vertical asymptotes. To find the vertical asymptotes of a rational function we set the denominator $Q(x)$ equal to 0 and solve for x. Those points where the denominator is 0 are undefined and like the reciprocal function will give us a vertical asymptote. Remember, the graph of the function will never cross vertical asymptotes.

4. **Other asymptotes:** Rational functions have one other asymptote. What does the function do as $x \to \pm\infty$? There are three possibilities for this asymptote. It can be the line $y = 0$, it can be the line $y = \frac{p}{q}$ where p is the leading coefficient of $P(x)$ and q is the leading coefficient of $Q(x)$, or it can be any polynomial curve (most of the functions that we will deal with will have a linear asymptote, but any polynomial is possible).

To find this asymptote we will use the principle that as the denominator of a fraction gets bigger the value of the fraction goes to 0. To apply this to a rational function we divide the denominator into the numerator and look at the remainder. As $x \to \pm\infty$ the remainder will go to 0 leaving the quotient as the asymptote.

A. If the degree of the denominator is larger than the degree of the numerator before dividing then the asymptote is $y = 0$.

B. If the degree of the denominator is equal to the degree of the numerator then the asymptote is $\frac{p}{q}$ where p is the leading coefficient of $P(x)$ and q is the leading coefficient of $Q(x)$.

C. If the degree of the numerator is larger than the degree of the denominator then the asymptote will be a polynomial. In this last case the degree of the asymptote will be $m - n$ where m is the degree of $P(x)$ and n the degree of $Q(x)$.

Because the function approaches this asymptote only as $x \to \pm\infty$, it is possible for the function to cross it. To find these intercepts we set $y = f(x)$ where y is the asymptote. Solving for x gives all intercepts. If the equation cannot be solved then there are no intercepts.

5. **Positive/negative regions:** Before we can graph the function we still need to know when the graph is positive (lies above the x axis) or negative (lies below the x axis). To find the positive and negative regions of a rational function we will use the principle that a quotient $\frac{a}{b}$ has the same sign as the product $\boldsymbol{ab}$. That is, dividing two numbers gives the same sign as multiplying them. What we wish to do here is find the signs of a quotient $\frac{P(x)}{Q(x)}$. Applying the above principle this quotient has the same signs as the product $P(x)Q(x)$. Because P and Q are both polynomials their product is also a polynomial. Graphing the polynomial $P(x)Q(x)$ with the techniques learned earlier, we can see where it is positive (above the x axis) or negative (below the x axis) by inspection. The rational function $\frac{P(x)}{Q(x)}$ will be positive or negative in the same regions.

CAUTION: The graph of $P(x)Q(x)$ has no relationship to the graph of $\frac{P(x)}{Q(x)}$ other than that they have the <u>same</u> sign. Do not try to mix the two graphs in any form.

Example 1. Graph the function $f(x) = \frac{x+1}{x-1}$.

1. x intercepts: Set the numerator equal to 0. This gives $x + 1 = 0$, or $x = -1$. This gives the x intercept (–1, 0).

2. y intercept: Evaluate $f(0) = \frac{0+1}{0-1} = \frac{1}{-1} = -1$. This gives the y intercept (0, –1).

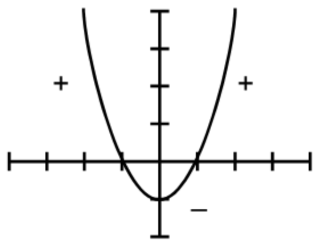
3. Vertical asymptotes: Set the denominator equal to 0. This gives $x - 1 = 0$, or $x = 1$. The vertical asymptote is $x = 1$.

4. Find asymptotes as $x \to \pm\infty$. Divide the function out and throw the remainder away (the fractional remainder goes to 0 as $x \to \pm\infty$).

$$\frac{x+1}{x-1} = 1 + \frac{2}{x-1}$$

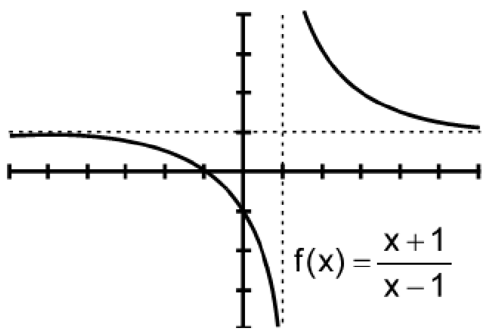
Setting the remainder equal to 0 gives the asymptote $y = 1$. To see if there are any intercepts with this asymptote we set $f(x) = y$ and solve for x. This gives us the equation $\frac{x+1}{x-1} = 1$. Multiplying out the denominator $x - 1 = x + 1$. Subtracting x from both sides –1 = 1. This equation cannot be solved, therefore there are no intercepts with this asymptote.

5. Sign changes: Multiply numerator and denominator together and graph the resulting polynomial. The two functions have the same sign. The polynomial is $(x-1)(x+1)$. This is a quadratic with x intercepts (–1, 0) and (1, 0). The graph is below.



y = (x + 1)(x – 1)

From the graph we see that the function is positive for $x > 1$ and $x < -1$ and is negative for $-1 < x < 1$.



Example 2. Graph the function $f(x) = \frac{x+1}{(x-1)(x+2)}$.

1. x intercept: $x + 1 = 0$ gives $x = -1$ or (–1, 0) for the x intercept.

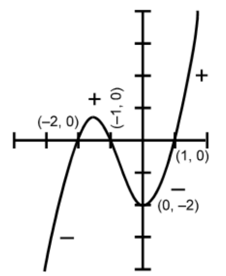
2. y intercept: $f(0) = \frac{0+1}{(0-1)(0+2)} = \frac{1}{(-1)(2)} = \frac{1}{-2} = -\frac{1}{2}$ or $\left(0, -\frac{1}{2}\right)$ for the y intercept.

3. Vertical asymptotes: Setting $(x-1)(x+2) = 0$ gives $x = 1$ and $x = -2$ for the vertical asymptotes.

4. Other asymptotes: The fraction $\frac{x+1}{(x-1)(x+2)}$ has numerator of lesser degree than the denominator. Letting $x \to \pm\infty$, the function $f(x) \to 0$. Therefore the other asymptote is

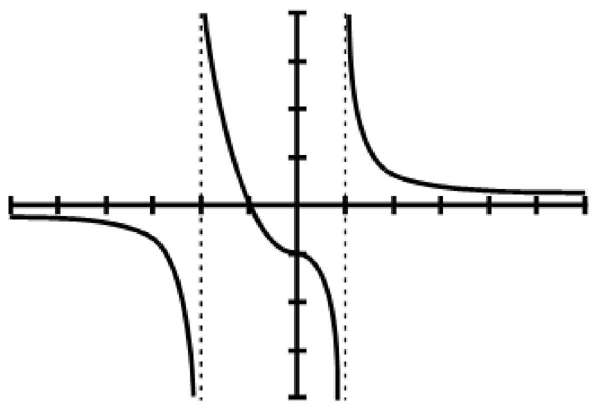
horizontal and is $y = 0$. We already know that the x intercept is the only intersection with this asymptote.

5. Sign changes: We graph the polynomial $y = (x-1)(x+1)(x+2)$. The graph is shown below.



y = (x – 1)(x + 1)(x + 2)

This gives negative for $x < -2$, positive for $-2 < x < -1$, negative for $-1 < x < 1$, and positive for $x > 1$.



$$f(x) = \frac{x+1}{(x-1)(x+2)}$$

Example 3. Graph the function $f(x) = \frac{x^2-x-6}{2x^2-x-3}$.

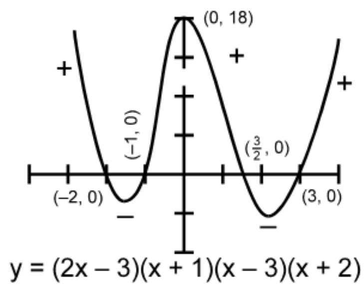
1. x intercepts: Setting the numerator equal to 0 and factoring gives $(x-3)(x+2) = 0$ or $x = 3$ and $x = -2$. The x intercepts are (–2, 0) and (3, 0).

2. y intercepts: Evaluating $f(0) = \frac{0^2-0-6}{2(0^2)-0-3} = \frac{-6}{-3} = 2$. The y intercept is (0, 2).

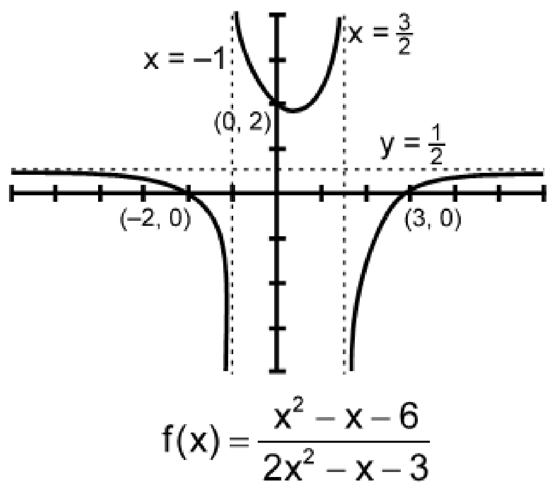
3. Vertical asymptotes: Setting the denominator equal to 0 gives $2x^2 - x - 3 = 0$, factoring and solving for x gives $(2x-3)(x+1) = 0$ or vertical asymptotes of $x = -1$ and $x = \frac{3}{2}$.

4. Other asymptotes: Dividing the fraction out gives $\frac{1}{2} - \frac{\frac{1}{2}(x+9)}{2x^2-x-3}$ Letting the fractional part go to 0 we have the asymptote $y = \frac{1}{2}$. We could also have found the asymptote by noticing that the numerator and denominator have the same degree. The ratio of the leading coefficients will then be the horizontal asymptote $\frac{1}{2}$.

5. Sign changes: We graph the polynomial $(2x-3)(x+1)(x-3)(x+2)$. This is shown below.



This gives positive for $x < -2$, negative for $-2 < x < -1$, positive for $-1 < x < \frac{3}{2}$, negative for $\frac{3}{2} < x < 3$, positive for $x > 3$



Example 4. Graph the function $f(x) = \frac{x^2-x-2}{x+2}$.

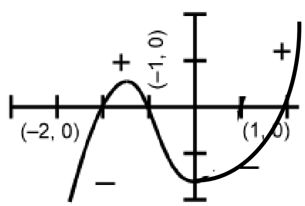
1. x intercepts: Setting the numerator equal to 0 and factoring gives $x^2 - x - 2 = (x+1)(x-2) = 0$. Our x intercepts are (–1, 0) and (2, 0).

2. y intercepts: Evaluating $f(0)$ gives $\frac{0^2-0-2}{0+2} = \frac{-2}{2} = -1$ for a y intercept of (0, –1).

3. Vertical asymptotes: Setting the denominator equal to 0 gives $x + 2 = 0$ or $x = -2$ for the vertical asymptote.

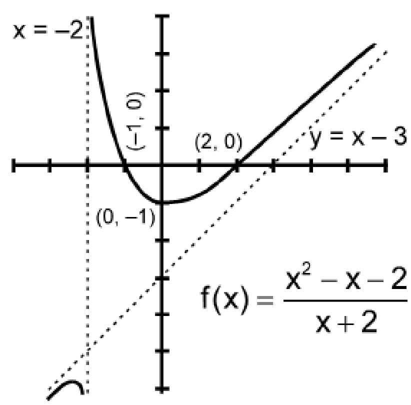
4. Other asymptotes: Dividing the denominator into the numerator gives $x - 3 + \frac{4}{x+2}$. Letting the fractional remainder go to 0 leaves us with the asymptote $y = x - 3$. To find any intercepts we set the function equal to the asymptote getting $\frac{x^2-x-2}{x+2} = x - 3$, multiplying the denominator out gives $x^2 - x - 2 = x^2 - x - 6$. Subtracting x^2 and adding x gives $-2 = -6$. This equation has no solutions therefore the function does not cross the asymptote $y = x - 3$.

5. Sign changes: To find the sign changes we graph the polynomial $y = (x^2 - x - 2)(x + 2)$ Factoring gives $y = (x + 1)(x - 2)(x + 2)$. This graph is below:



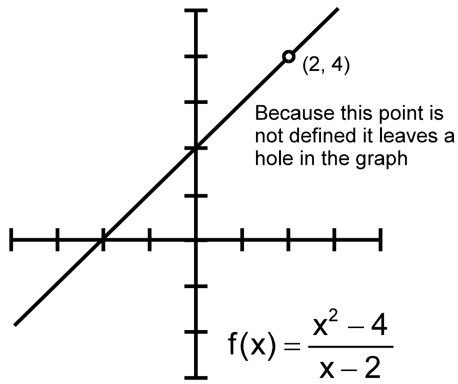
y = (x + 1)(x – 2)(x + 2)

This gives negative for $x < -2$, positive for $-2 < x < -1$, negative for $-1 < x < 2$, and positive for $x > 2$



Example 5. Graph the function $f(x) = \frac{x^2-4}{x-2}$.

If you look carefully at the numerator you will notice that it factors into $(x+2)(x-2)$. The denominator is $(x-2)$ which will divide out with the $(x-2)$ in the numerator provided that $x \neq 2$. If $x = 2$, the function is not defined because $f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$ which is undefined. As long as $x \neq 2$, $f(x) = x + 2$.



3.5 Graphs of Rational Functions Practice

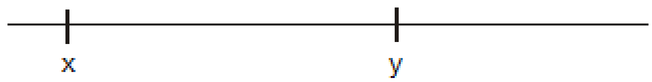
Graph each of the following rational functions:

1. $\frac{x+2}{x-1}$
2. $\frac{x-2}{x+1}$
3. $\frac{x+3}{x+4}$
4. $\frac{x+4}{(x+3)(x+1)}$
5. $\frac{x-4}{(x+3)(x+1)}$
6. $\frac{x+2}{x^2-9}$
7. $\frac{x+3}{x^2-4}$
8. $\frac{2x}{x+3}$
9. $\frac{2x}{x-3}$
10. $\frac{x+2}{2x+3}$
11. $\frac{x-2}{2x-3}$
12. $\frac{5x-9}{x+3}$
13. $\frac{4x+9}{x-3}$
14. $\frac{x-3}{x-3}$
15. $\frac{x+4}{x+4}$
16. $\frac{x-3}{x^2-9}$
17. $\frac{x+2}{x^2-9}$
18. $\frac{x^2-4}{x}$
19. $\frac{x^2-9}{x}$
20. $\frac{x+1}{x}$
21. $\frac{x-2}{x^2}$
22. $\frac{x-4}{x^2-4}$
23. $\frac{x+4}{x^2-9}$
24. $\frac{x-2}{x^2-25}$
25. $\frac{x+3}{x^2-25}$
26. $\frac{x+2}{x^2+5x+4}$
27. $\frac{x-2}{x^2-6x+5}$
28. $\frac{x}{(x-2)(x+1)(x-4)}$
29. $\frac{x+4}{(x-1)(x-3)(x+5)}$
30. $\frac{x^2+2x+5}{x+1}$
31. $\frac{x^2+2x+5}{x-1}$
32. $\frac{x^2-5x+6}{x^2-16}$
33. $\frac{x^2-7x+6}{(x+1)(x+4)(x-2)}$
34. $\frac{-2x^2+7x-5}{x^2-x-12}$
35. $\frac{(3-x)(2-x)(1-x)}{(x+3)(x+2)(x+1)}$
36. $\frac{(1+x)(2-x)(2+x)}{(3-x)(4+x)(x-1)}$
37. $\frac{(x-1)(x+2)}{3-x}$
38. $\frac{(x^2-6x+8)(x-1)}{9+6x+x^2}$
39. $\frac{x^3-x}{2+x}$
40. $\frac{2-3x+x^2}{(x-1)(x-3)(x+5)}$
41. $\frac{2-3x+x^2}{(x+1)(4+x)(x-2)}$
42. $\frac{x^2+2x+1}{x^3-3x^2+3x-1}$

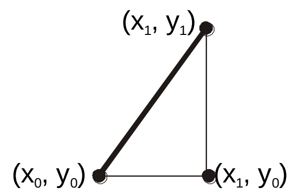
3.6 Midpoint, Distance, and Circles

Distance

Given two points on a plane it is useful and frequently necessary to find the distance between the two points and the midpoint or the point on the line determined by the two points and halfway between them. Consider two points x and y on the number line.



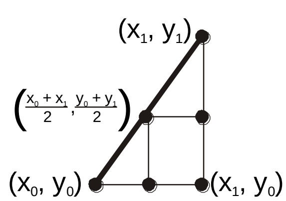
The distance between these two points is simply the difference of the larger minus the smaller. If $x = 4$ and $y = 12$ then the distance between them is $12 - 4$ or 8. Also, it doesn't matter whether the line is horizontal or vertical or any angle in between. The distance between the two points is still the larger minus the smaller.



Consider the points (x_0, y_0) and (x_1, y_1) on the plane and the line segment that connects them. We want to find the length of the line segment. Extend the horizontal and vertical lines from the two points as shown in the picture to form a right triangle. We know from the Pythagorean theorem that the length of the hypotenuse is the square root of the sums of the squares of the legs of the triangle. The base of the triangle is a horizontal line with length $x_1 - x_0$ and the vertical distance is $y_1 - y_0$. Using Pythagoras we get the distance between the two points, $D = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$.

Midpoint

The midpoint between two points on a straight line is the average of the value of the points. So on the horizontal line above the midpoint is $\frac{x+y}{2}$.



On the two points in a plane the midpoint of the hypotenuse has to have an x coordinate that is halfway between the x values and a y coordinate that is halfway between the y coordinates. Or, the midpoint is $\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right)$.

Example 1. Find the midpoint and distance between the two points (4, –2) and (–8, 7).

$(4,-2),(-8,7)$ Find the midpoint using $\left(\frac{x_0+x_1}{2},\frac{y_0+y_1}{2}\right)$

$$\left(\frac{4+(-8)}{2},\frac{-2+7}{2}\right)$$ Simplify

$$\left(-2,\frac{5}{2}\right)$$ Final answer for midpoint

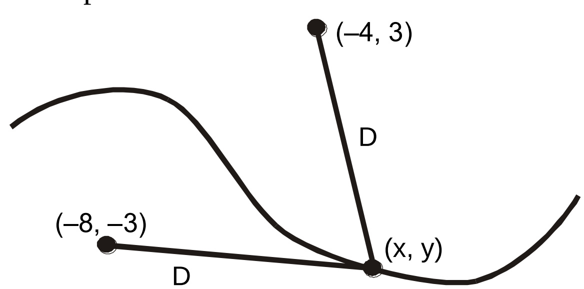
$(4,-2),(-8,7)$ Find distance using $D=\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$

$$D=\sqrt{(-8-4)^2+\big(7-(-2)\big)^2}$$ Simplify

$$D=\sqrt{(-12)^2+(9)^2}$$
$$D=\sqrt{144+81}$$
$$D=\sqrt{225}$$
$$D=15$$ Final answer for distance

Example 2: Find the locus of all points equidistant to the two points (–8, –3) and (–4, 3).

A locus of points is simply the curve formed by the points in a plane. In this case we want the curve where each point on the curve is the same distance from the two given points. Notice we are talking about each point not all points. The problem is not possible if all points on the curve are the same distance from the given two points. Let the point (x,y) be on the curve. Draw the line that connects the two points (–4, 3) and (–8, –3) to the point (x,y). The distance D between the two points and the point (x,y) is the same. We need to find the function $y=f(x)$ that represents the curve.



We know that the distance between each of the points and (x,y) can be found using the distance formula. We do this for both points:

$$D=\sqrt{(x+4)^2+(y-3)^2}$$
$$D=\sqrt{(x+8)^2+(y+3)^2}$$ Both distances are the same, set them equal

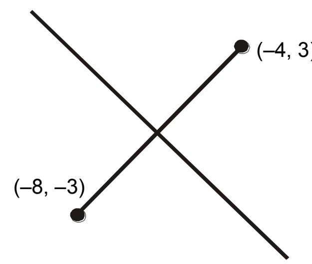
$$\sqrt{(x+4)^2+(y-3)^2}=\sqrt{(x+8)^2+(y+3)^2}$$ Square both sides, eliminate the radical

$$(x+4)^2+(y-3)^2=(x+8)^2+(y+3)^2$$ Square binomials and subtract squared terms

$$x^2 + 8x + 16 + y^2 - 6y + 9 = x^2 + 16x + 64 + y^2 + 6y + 9$$

$8x + 16 - 6y + 9 = 16x + 64 + 6y + 9$	Combine like terms
$8x - 6y + 25 = 16x + 6y + 73$	Solve for y, subtract $8x$, 25, and $6y$
$-12y = 8x + 48$	Divide by -12
$y = -\frac{2}{3}x - 4$	Final answer

Example 3. Find the perpendicular bisector of the line segment that connects the points $(-8, -3)$ and (–4, 3).



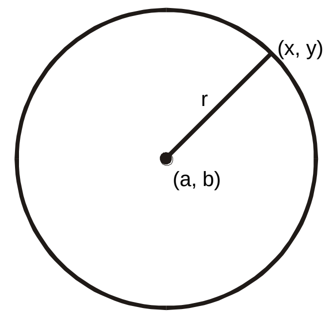
To find the perpendicular bisector we need to know the midpoint between the two points and the slope of the line connecting the two points. We know that a perpendicular line has a negative reciprocal slope. With point and slope we can find the equation of the line. First we find the midpoint using the midpoint formula.

$\left(\frac{-8 + (-4)}{2}, \frac{-3 + 3}{2}\right)$	Simplify to get point on the perpendicular bisector
$(-6,0)$	Find the slope connecting the points
$\frac{3 - (-3)}{-4 - (-8)} = \frac{6}{4} = \frac{3}{2}$	Find the slope of the perpendicular line
$m = -\frac{2}{3}$	Put point and slope into $y = mx + b$
$0 = -\frac{2}{3}(-6) + b$	Multiply
$0 = 4 + b$	Subtract 4 from both sides
$-4 = b$	Give the equation with slope and intercept
$y = -\frac{2}{3}x - 4$	Final answer

Not surprisingly, it is the same equation we derived in example 2.

Equation of a Circle

We want to find the equation of all points equidistant from a fixed point. Geometrically this is a circle of radius r where the fixed point is the center of the circle. Let (x, y) be a point on the circumference of a circle of radius r and center at (a, b). Then the distance between the center and the circumference is $r = \sqrt{(x-a)^2 + (y-b)^2}$ or $r^2 = (x-a)^2 + (y-b)^2$. This is the equation of a circle of radius r and centered at the point (a, b).



Example 4: Find the equation of a circle with radius 3 and centered at the point (–2, 5).

Radius: 3 Center: $(-2,5)$	Put values into equation of circle
$3^2 = (x+2)^2 + (y-5)^2$	Simplify left side
$9 = (x+2)^2 + (y-5)^2$	Final answer

Example 5: Find the center and radius of the circle $(x-3)^2 + (y+1)^2 = 25$.

$(x-3)^2 + (y+1)^2 = 25$	From formula, center is opposite of a and b
Center: $(3, -1)$	The 25 is r^2
$r^2 = 25$	Solve, take the square root
$r = 5$	Final answer

Example 6: Find the center and radius of the circle $2x^2 - 6x + 2y^2 + 2y - 27 = 0$.

This problem is slightly different than the previous one. This one is not in the standard form $r^2 = (x-a)^2 + (y-b)^2$. This means that we need to complete the square on the equation to put it into squared form.

$2x^2 - 6x + 2y^2 + 2y - 27 = 0$	Add 27 to both sides of the equation
$2x^2 - 6x + 2y^2 + 2y = 27$	Divide both sides by 2
$x^2 - 3x + y^2 + y = \frac{27}{2}$	Complete the square on x and y. Need to add half of the second term squared
For x: $\left(\frac{1}{2}\cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ For y: $\left(\frac{1}{2}\cdot 1\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$	Add to both sides of equations
$\left(x^2 - 3x + \frac{9}{4}\right) + \left(y^2 + y + \frac{1}{4}\right) = \frac{27}{2} + \frac{9}{4} + \frac{1}{4}$	Factor and simplify
$\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 16$	From formula, identify center and radius
Center: $\left(\frac{3}{2}, -\frac{1}{2}\right)$ Radius = 4	Final answer

3.6 Midpoint, Distance, and Circles Practice

Find the distance and midpoint of each of the following pairs of points:

1. $(1,1)$ and $(9,7)$
2. $(1,12)$ and $(6,0)$
3. $(-4,10)$ and $(4,-5)$
4. $(-7,-4)$ and $(2,8)$
5. $(-1,2)$ and $(5,4)$
6. $(2,10)$ and $(10,2)$
7. $\left(\frac{1}{2},1\right)$ and $\left(-\frac{5}{2},\frac{4}{3}\right)$
8. $\left(-\frac{1}{3},-\frac{1}{3}\right)$ and $\left(-\frac{1}{6},-\frac{1}{2}\right)$
9. $(-36,-18)$ and $(48,-72)$

(Hint for 10-12 use distance formula, then find min using vertex formula)

10. What point is nearest to (3, 0) on the curve $y = \sqrt{x}$?

11. What point is closest to (4, 1) on the curve $y = \sqrt{x-2} + 1$?

12. Find the coordinates of the point on the line $y = 3x + 1$ closest to (4, 0).

13. Find x so the distance between the points is 13.

    a. $(1,2)$ and $(x,-10)$
    b. $(-8,0)$ and $(x,5)$

14. Find y so the distance between the points is 17.

    a. $(0,0)$ and $(8,y)$
    b. $(-8,4)$ and $(7,y)$

15. Find a relationship between x and y so that (x,y) is equidistant from the two points.

    a. $(4,-1)$ and $(-2,3)$
    b. $\left(3,\frac{5}{2}\right)$ and $(-7,1)$

16. Find the perpendicular bisector of the line that connects the two points

    a. $(1,1)$ and $(-1,3)$
    b. $(-4,2)$ and $(-1,3)$

Find the standard form of the equation for each of the following circles:

17. Center $(0,0)$; radius $= 3$

18. Center $(0,0)$; radius $= 5$

19. Center $(2,-1)$; radius $= 4$

20. Center $\left(0,\frac{1}{3}\right)$; radius $= \frac{1}{3}$

21. Center $(-1,2)$ passing through $(0,0)$

22. Center $(3,-2)$, passing through $(-1,1)$

23. Endpoints of diameter $(0,0)$ & $(6,8)$

24. Endpoints of diameter $(-4,-1)$ & $(4,1)$

Find the center and radius of each of the following circles:

25. $x^2 + y^2 - 2x + 6y + 6 = 0$

26. $x^2 + 2x + y^2 + 6y - 15 = 0$

27. $x^2 + y^2 - 2x + 6y + 10 = 0$

28. $3x^2 + 3y^2 - 6y - 1 = 0$

29. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$

30. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$

31. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$

32. $x^2 + y^2 - 4x + 2y + 3 = 0$

Chapter 4

Exponents and Logarithms

4.1 Exponential Equations with Common Base

Definition:

$$a^x = a^y \quad \leftrightarrow \quad x = y$$

Example 1: Solve

$5^x = 5^{13}$	By direct application of above definition
$x = 13$	Final answer

Example 2: Solve

$6^x = 216$	Make both sides powers of the same number
$6^x = 6^3$	Set exponents equal
$x = 3$	Final answer

Example 3: Solve

$4^x = 8$	Make both sides powers of the same number
$(2^2)^x = 2^3$	Multiply exponents
$2^{2x} = 2^3$	Set exponents equal
$2x = 3$	Solve for x
$x = \frac{3}{2}$	Final answer

Example 4: Solve

$27^x = 3$	Make both sides powers of the same number
$(3^3)^x = 3^1$	Multiply exponents
$3^{3x} = 3^1$	Set exponents equal
$3x = 1$	Solve for x
$x = \frac{1}{3}$	Final answer

Example 5: Solve

$9^x = \frac{1}{3}$	Negative exponents create fractions
$(3^2)^x = 3^{-1}$	Multiply exponents
$3^{2x} = 3^{-1}$	Set exponents equal
$2x = -1$	Solve for x
$x = -\frac{1}{2}$	Final answer

Example 6: Solve

$125^{3x-7} = \frac{1}{25}$	Make both sides powers of the same number
$(5^3)^{3x-7} = 5^{-2}$	Put binomial in parentheses
$5^{3(3x-7)} = 5^{-2}$	Set exponents equal
$3(3x-7) = -2$	Distribute
$9x - 21 = -2$	Add 21
$9x = 19$	Divide by 9
$x = \frac{19}{9}$	Final answer

Example 7: Solve

$\left(\frac{4}{9}\right)^x = \frac{27}{8}$	Make both sides power of the same number
$\left[\left(\frac{2}{3}\right)^2\right]^x = \left(\frac{3}{2}\right)^3$	Because the fractions are reciprocals of each other, we flip one of them by making the exponent negative.
$\left[\left(\frac{2}{3}\right)^2\right]^x = \left(\frac{2}{3}\right)^{-3}$	Multiply exponents
$\left(\frac{2}{3}\right)^{2x} = \left(\frac{2}{3}\right)^{-3}$	Set exponents equal
$2x = -3$	Solve for x
$x = -\frac{3}{2}$	Final answer

Example 8: Solve

$8^{3x-1} = 4^{4x+3}$	Make both sides powers of the same number
$(2^3)^{3x-1} = (2^2)^{4x+3}$	Put binomials in parentheses
$2^{3(3x-1)} = 2^{2(4x+3)}$	Set exponents equal
$3(3x-1) = 2(4x+3)$	Distribute
$9x - 3 = 8x + 6$	Subtract $8x$, add 3 to both sides
$x = 9$	Final answer

Example 9: Solve

$5^{x^2} = 25^x \cdot 125$	Make both sides power of the same number
$5^{x^2} = (5^2)^x(5^3)$	Multiply exponents
$5^{x^2} = 5^{2x} \cdot 5^3$	Add exponents on left
$5^{x^2} = 5^{2x+3}$	Set exponents equal
$x^2 = 2x + 3$	Because we have a squared exponents, set equal to zero
$x^2 - 2x - 3 = 0$	Factor
$(x-3)(x+1) = 0$	Set each factor equal to zero
$(x-3) = 0$ or $x + 1 = 0$	Solve both equation
$x = 3, -1$	Final answer

Example 10: Solve

$3^x + 9 \cdot 3^{-x} = 10$	Clear negative exponent, multiply by 3^x
$(3^x)^2 + 9 = 10 \cdot 3^x$	This is quadratic in form, set equal to zero
$(3^x)^2 - 10 \cdot (3^x) + 9 = 0$	Factor
$(3^x - 9)(3^x - 1) = 0$	Set each factor equal to zero
$3^x - 9 = 0$ or $3^x - 1 = 0$	Add to both sides
$3^x = 9$ or $3^x = 1$	Make both sides power of the same number
$3^x = 3^2$ or $3^x = 3^0$	Set exponents equal
$x = 2,0$	Final answer

Example 11: Solve

$3^{5-x} = 9^{x^2+1}$	Make both sides power of the same number
$3^{5-x} = (3^2)^{x^2+1}$	Put binomial in parentheses
$3^{5-x} = 3^{2(x^2+1)}$	Set the exponents equal
$5 - x = 2(x^2 + 1)$	Distribute
$5 - x = 2x^2 + 2$	Set equation equal to zero
$0 = 2x^2 + x - 3$	Factor
$0 = (2x + 3)(x - 1)$	Set each factor equal to zero
$2x + 3 = 0$ or $x - 1 = 0$	Solve each equation
$x = -\frac{3}{2}, 1$	Final answer

Example 12: Solve

$49^{x^2-2} - 8 \cdot 7^{x^2-2} + 7 = 0$	Write 49 as 7^2 to get quadratic form
$(7^2)^{x^2-2} - 8 \cdot 7^{x^2-2} + 7 = 0$	Rewrite $(7^2)^{x^2-2}$ as $\left(7^{x^2-2}\right)^2$
$\left(7^{x^2-2}\right)^2 - 8 \cdot 7^{x^2-2} + 7 = 0$	Factor
$\left(7^{x^2-2} - 7\right)\left(7^{x^2-2} - 1\right) = 0$	Set each factor equal to zero
$7^{x^2-2} - 7 = 0$ or $7^{x^2-2} - 1 = 0$	Add
$7^{x^2-2} = 7$ or $7^{x^2-2} = 1$	Make both sides power of the same number
$7^{x^2-2} = 7^1$ or $7^{x^2-2} = 7^0$	Set exponents equal
$x^2 - 2 = 1$ or $x^2 - 2 = 0$	Add 2 to both sides of each
$x^2 = 3$ or $x^2 = 2$	Square root both sides
$x = \pm\sqrt{3}, \pm\sqrt{2}$	Final answer

4.1 Exponential Equations with Common Base Practice

Solve for the variable.

1. $2^x = 8$
2. $3^x = 27$
3. $5^x = 125$
4. $3^x = 243$
5. $9^x = 27$
6. $8^x = 2$
7. $343 = 7^x$
8. $625 = 5^x$
9. $4 = 8^x$
10. $27^x = 9$
11. $81^x = 27$
12. $125^x = 25$
13. $b^3 = 343$
14. $b^5 = 32$
15. $a^3 = 1331$
16. $3125 = a^5$
17. $25^x = \frac{1}{5}$
18. $27^x = \frac{1}{3}$
19. $8^{2x-2} = \frac{1}{16}$
20. $9^{3x-2} = \frac{1}{27}$
21. $\left(\frac{2}{3}\right)^x = \frac{27}{8}$
22. $\left(\frac{10}{7}\right)^x = 0.49$
23. $\left(\frac{4}{9}\right)^x = 1.5$
24. $\left(\frac{64}{25}\right)^x = 1.6$
25. $9^{y+1} = 243^{1-2y}$
26. $125^{2-3y} = 625^{y-3}$
27. $\left(\frac{4}{5}\right)^{x+3} = \frac{25}{16}$
28. $\left(\frac{7}{2}\right)^x = \frac{8}{343}$
29. $3^{x+1} + 3 = 30$
30. $4^{2x} = 8^{3x-4}$
31. $3^{x^2+4x} = \frac{1}{27}$
32. $3^{5x} \cdot 9^{x^2} = 27$
33. $4^{x^2} \cdot 2^x = 2$
34. $4^{x^2} \cdot 4^x = 16^3$
35. $3^{x^2} \cdot 81^x = \frac{1}{27}$
36. $25^{x^2} \cdot 5^x = 125$
37. $\frac{27^{x^2}}{81^x} = \frac{1}{3}$
38. $\frac{2^{x^2}}{8^x} = \frac{1}{4}$
39. $\frac{8^{x^2}}{128^x} = \frac{1}{4}$
40. $\frac{27^{3x^2}}{27^x} = 9$
41. $2^{2x-3} = 4^{x^2-3x-2}$
42. $2^{x+1} + 2^{-x} = 3$
43. $5^{2x-3} = 25^{x^2-3x-2}$
44. $7^{x+2} - \frac{1}{7} \cdot 7^{x-1} - 14 \cdot 7^{x-1} + 2 \cdot 7^x = 0$
45. $4^{x^2+2} - 9 \cdot 2^{x^2+2} + 8 = 0$

4.2 Properties of Logarithms

Definitions and Properties

Logarithm: The inverse of an exponential function. The inverse of $f(x) = a^x$ is $f^{-1}(x) = \log_a x$. Logarithms tell us the exponential "size" of a number.

When we want to find $\log_a b$ we are asking "a to what power equals b?"

As exponential functions and logarithmic functions are inverses, the domain of the exponential function is the range of the logarithmic function. Similarly, the range of the exponential function is the domain of the logarithmic function.

Function	Domain	Range
Exponential Function	$(-\infty, \infty)$	$(0, \infty)$
Logarithmic Function	$(0, \infty)$	$(-\infty, \infty)$

You should be able to identify log equivalencies: $\log_a b = x \leftrightarrow a^x = b$

Example 1: Convert to logarithmic form

$5^x = 14$ Identify base of 5 and exponent of x

$\log_5 14 = x$ Final answer

Example 2: Convert to exponential form

$\log_{10} x = 6$ Identify base of 10 and exponent of 6

$10^6 = x$ Final answer

Example 3: Solve

$\log_5 x = 3$ Rewrite as exponent

$5^3 = x$ Simplify

$125 = x$ Final answer

Example 4: Solve

$\log_2(13x-2)=7$	Convert to exponent
$2^7=13x-2$	Evaluate exponent
$128=13x-2$	Add 2
$130=13x$	Divide by 13
$10=x$	Final answer

Example 5: Evaluate

$\log_2(\log_3 81)$	Work from inside out
$\log_3 81$	"3 to what power is 81?"
4	Now work on the outside log
$\log_2 4$	"2 to what power is 4?"
2	Final answer

Example 6: Evaluate

$\log_{\sqrt{3}}\left(\frac{1}{27}\right)$	Set equal to x
$x=\log_{\sqrt{3}}\left(\frac{1}{27}\right)$	Rewrite as exponent
$\sqrt{3}^x=\frac{1}{27}$	Make both sides powers of the same number
$\left(3^{1/2}\right)^x=3^{-3}$	Multiply exponents
$3^{x/2}=3^{-3}$	Set exponents equal
$\frac{x}{2}=-3$	Multiply both sides by 2
$x=-6$	Final answer

Properties of Logarithms

From the properties of exponents we can find some important properties of logarithms:

$$\log_a 1 = 0$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^r = r \log_a x$$

Example 7: Solve

$\log_e e^7 = x$ The log and the base are inverses, leaving just the exponent

$7 = x$ Final answer

Example 8: Write as a single logarithm

$\log_5 x + \log_5 y - \log_5 13$ Use properties of addition and subtraction

$\log_5 \frac{xy}{13}$ Final answer

Generally speaking, we can put the arguments of positive logarithms in the numerator and the arguments of negative logarithms in the denominator.

Example 9: Write as a single logarithm

$3\log_7 z - 2\log_7 x + \log_7 y - \log_7 t$ Use exponent property to move coefficients into logs

$\log_7 z^3 - \log_7 x^2 + \log_7 y - \log_7 t$ Positive logarithms in numerator and negative logarithms in denominator

$\log_7 \frac{yz^3}{tx^2}$ Final answer

Log properties can be used to evaluate unknown logs based on known logs of the same base. This is illustrated in the following examples:

Example 10: Suppose $\log_b 2 = 0.3562$ and $\log_b 3 = 0.5646$. Find $\log_b 6$

$\log_b 6$	Rewrite 6 using 2's and 3's
$\log_b(2 \cdot 3)$	Expand logarithm
$\log_b 2 + \log_b 3$	Replace logs with their values given
$0.3562 + 0.5646$	Add
0.9208	Final answer

Example 11: Suppose $\log_b 2 = 0.3562$ and $\log_b 3 = 0.5646$. Find $\log_b \frac{2}{3} b^4$

$\log_b \left(\frac{2b^4}{3}\right)$	Expand the logarithm
$\log_b 2 + \log_b b^4 - \log_b 3$	Replace logs with given values, $\log_b b^4 = 4$
$0.3562 + 4 - 0.5646$	Add and subtract
3.7916	Final answer

Change of Base Theorem:

For any positive c such that $c \neq 1$ we have:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example 12: Use the change of base theorem to write $\log_9 x$ as a logarithm with a base of 5

$\log_9 x$	Using change of base formula
$\frac{\log_5 x}{\log_5 9}$	Final answer

On your calculator, you have two logarithm buttons: $\log x = \log_{10} x$ (common logarithm) and $\ln x = \log_e x$ (natural logarithm; the abbreviation comes from the Latin "*logaritmus naturalis*")

4.2 Properties of Logarithms Practice

Convert to an exponential equation

1. $t = \log_4 7$
2. $\log_5 5 = 1$
3. $\log_{10} 7 = 0.845$
4. $\log_a 10 = 2.3036$
5. $\log_a 0.38 = -0.9676$
6. $\log_a W = -2$
7. $h = \log_6 29$
8. $\log_{10} 0.1 = -1$

Convert to a logarithmic equation

9. $10^{0.4771} = 3$
10. $t^k = Q$
11. $a^{-0.0987} = 0.906$
12. $2^5 = 32$
13. $10^{-2} = 0.01$
14. $m^a = P$
15. $r^{-x} = M$

Solve

16. $\log_{10} x = 3$
17. $\log_5\left(\frac{1}{25}\right) = x$
18. $\log_x 16 = 2$
19. $\log_3 x = -2$
20. $\log_2 16 = x$
21. $\log_3 x = 2$
22. $\log_x 64 = 3$
23. $\log_8 x = \frac{1}{3}$
24. $\log_3 3 = x$
25. $\log_4 x = 3$
26. $\log_2 x = -1$
27. $\log_{32} x = \frac{1}{5}$
28. $|\log_3 x| = 3$
29. $\log_{\sqrt{5}} x = -3$
30. $\log_{\sqrt{125}} x = \frac{2}{3}$
31. $\log_b b^{2x^2} = x$
32. $\log_x \sqrt[5]{36} = \frac{1}{10}$
33. $\log_\pi \pi^4 = x$
34. $\log_4(3x-2) = 2$
35. $\log_9(x^2+2x) = \frac{1}{2}$

Evaluate

36. $\log_{\frac{1}{4}}\left(\frac{1}{64}\right)$
37. $\log_2(\log_2 256)$
38. $\log_{\sqrt{3}}\left(\frac{1}{81}\right)$
39. $\log_4(\log_3 81)$
40. $\log_{\frac{1}{5}} 25$

Expand into several logarithms

41. $\log_b PQ$

42. $\log_b \left(\frac{P}{Q}\right)$

43. $\log_a \sqrt{\frac{z^3}{xy}}$

44. $\log_a \left(\frac{x^2}{y^3 a}\right)$

45. $\log_m \left(\frac{a^3 b^4}{m^5 n^9}\right)$

46. $\log_a \left(\frac{p^3 q^2}{z^4}\right)$

47. $\log_a \sqrt[4]{\frac{ab}{c^3}}$

48. $\log_b \left(\frac{ab^5}{m^3 n^4}\right)$

49. $\log_a \sqrt{4 - x^2}$

50. $\log_a \frac{x - y}{\sqrt{x^2 - y^2}}$

51. $\ln \frac{x^2 - 1}{x^3}$

52. $\ln \sqrt{x^2(x + 2)}$

Express as a single logarithm

53. $\log C + \log A + \log B + \log I + \log N$

54. $\log_2 x - \log_2 25$

55. $5 \log_a x - \log_a y + \frac{1}{4} \log_a z$

56. $\frac{1}{2} \log_a x - 7 \log_a y + \log_a z$

57. $\log_a \left(\frac{\sqrt{x}}{b}\right) - \log_a \sqrt{bx}$

58. $\log_a \left(\frac{b}{\sqrt{x}}\right) + \log_a \sqrt{bx}$

59. $\frac{2}{3} \log_a x - \frac{1}{3} \log_a y$

60. $\frac{1}{2} \log_a x + 4 \log_a y - 3 \log_a x$

61. $\log_a 2x + 3(\log_a x - \log_a y)$

62. $\log_a x^2 - \log_a \sqrt{x}$

63. $\log_a \left(\frac{a}{\sqrt{x}}\right) - \log_a \sqrt{ax}$

64. $\log_a(x^2 - 4) - \log_a(x - 2)$

65. $\frac{1}{3}[2 \ln(x + 1) + \ln x - \ln(x^2 - 1)]$

66. $2[\ln(x + 1) + \ln(x - 1)] - 3 \ln(x^2 - 1)$

Use the change of base formula to rewrite the logarithm with the given base

67. $\log_3 5$ in base 10

68. $\log_5 3$ in base e

69. $\log_2 x$ in base 10

70. $\log_x y$ in base y

71. $\log_4 8$ in base 2

72. $\log_{\frac{1}{5}} 10$ in base 5

Evaluate, given that $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$

73. $\log_b 6$

74. $\log_b \left(\frac{3}{2}\right)$

75. $\log_b 25$

76. $\log_b \sqrt{2}$

77. $\log_b \left(\frac{1}{4}\right)$

78. $\log_b \sqrt{5b}$

79. $\log_b \left(\frac{4.5^3}{\sqrt{3}}\right)$

80. $\log_b 15$

81. $\log_b \left(\frac{5}{3}\right)$

82. $\log_b 18$

83. $\log_b \left(\frac{9}{2}\right)$

84. $\log_b \sqrt[3]{75}$

85. $\log_b (3b^2)$

86. $\log_b 1$

4.3 Exponential Equations with Different Bases

Generally speaking, we solve exponential equations with different bases by applying a logarithm to both sides of an equation and using properties to simplify. We usually use $\log x$ or $\ln x$ so as to making using a calculator easier.

Example 1: Solve

$7^x = 3$	Take a natural log of both sides
$\ln 7^x = \ln 3$	Exponent moves to front
$x \ln 7 = \ln 3$	Divide by $\ln 7$
$x = \dfrac{\ln 3}{\ln 7}$	Type into calculator
$x \approx 0.5646$	Final answer

Example 2: Solve

$5^{x+4} = 15$	Take a natural log of both sides
$\ln 5^{x+4} = \ln 15$	Treat exponent as one whole unit, move to front
$(x+4)\ln 5 = \ln 15$	Divide by $\ln 5$
$x + 4 = \dfrac{\ln 15}{\ln 5}$	Subtract 4
$x = \dfrac{\ln 15}{\ln 5} - 4$	Type into calculator
$x \approx -2.3174$	Final answer

Example 3: Solve

$e^x = 9$	Take a natural log of both sides
$\ln e^x = \ln 9$	Natural log and base e are inverses, leaving just the exponent
$x = \ln 9$	Type into calculator
$x \approx 2.1972$	Final answer

Example 4: Solve

$10^x = 7$	Take *common* log of both sides (base 10)
$\log 10^x = \log 7$	Common log and base 10 are inverses, leaving just the exponent
$x = \log 7$	Type into calculator
$x \approx 0.8451$	Final answer

Example 5: Solve

$4^x = 5^x$	Take a natural log of both sides
$\ln 4^x = \ln 5^x$	Exponents move to front
$x \ln 4 = x \ln 5$	Subtract $x \ln 5$ from both sides
$x \ln 4 - x \ln 5 = 0$	Factor out the x
$x(\ln 4 - \ln 5) = 0$	Divide by the binomial
$x = \frac{0}{\ln 4 - \ln 5}$	Simplify
$x = 0$	Final answer

In the above example, it is important to note that we cannot divide by x, should x be zero, as it turned out to be, dividing by zero is undefined and therefore not permissible.

Example 6: Solve

$2^{3x} = 4^{5x-6}$	Take a natural log of both sides
$\ln 2^{3x} = \ln 4^{5x-6}$	Exponents move to front
$3x \ln 2 = (5x-6) \ln 4$	Distribute
$3x \ln 2 = 5x \ln 4 - 6 \ln 4$	Subtract term with x to other sides
$3x \ln 2 - 5x \ln 4 = -6 \ln 4$	Factor out the x
$x(3 \ln 2 - 5 \ln 4) = -6 \ln 4$	Divide by the binomial
$x = \dfrac{-6 \ln 4}{3 \ln 2 - 5 \ln 4}$	Type into calculator
$x \approx 1.7143$	Final answer

4.3 Exponential Equations with Different Bases Practice

Solve for the variable

1.	$3^x = 5$	2.	$4^x = 9$	3.	$e^x = 21$
4.	$10^x = 16$	5.	$8^{x+3} = 6$	6.	$12^{x-7} = 2$
7.	$e^{3x} = 7$	8.	$10^{4x-5} = 2$	9.	$5^x = 3^x$
10.	$10^x = e^x$	11.	$10^{x+2} = e^{x+1}$	12.	$e^{4x} = 10^{x-2}$
13.	$2^{x+1} = 3^{x-1}$	14.	$3^{2x} = 2^{x-1}$	15.	$e^{x^2} = 2^x$
16.	$2^{x^2-1} = 3^{x+1}$	17.	$5^{x-1} = 3^{x^2-x-5}$	18.	$7^{x^2+2x-1} = 3^{2-x^2}$

4.4 Solving Equations with Logarithms

We generally want to have as few logarithms on each side of an equation as possible. To get rid of logarithms, we make both sides powers of their base or use log equivalence and convert to exponents. Don't forget to check your answers when you finish as logarithms cannot have negative arguments (Domain is $x > 0$). These answers must be excluded from your final solution.

Example 1: Solve

$\log_{12} x + \log_{12}(x+1) = 1$	Combine to a single log
$\log_{12} x(x+1) = 1$	Convert to exponent
$12^1 = x(x+1)$	Evaluate 12^1 and distribute on right
$12 = x^2 + x$	Subtract 12 from both sides
$0 = x^2 + x - 12$	Factor
$0 = (x+4)(x-3)$	Set each factor equal to zero
$x + 4 = 0$ or $x - 3 = 0$	Solve both equations
$x = -4, 3$	Check if this makes arguments of logs negative
$\log_{12} -4$ and $\log_{12}(-4+1)$ are undefined $\log_{12} 3$ and $\log_{12}(3+1)$ are ok	Exclude -4 from solution
$x = 3$	Final answer

Example 2: Solve

$\log_7(x+3) - \log_7 x = 2$	Combine to a single log
$\log_7 \dfrac{x+3}{x} = 2$	Convert to exponent
$7^2 = \dfrac{x+3}{x}$	Evaluate exponent
$49 = \dfrac{x+3}{x}$	Multiply by denominator
$49x = x + 3$	Subtract x from both sides
$48x = 3$	Divide both sides by 48
$x = \dfrac{1}{16}$	Check if this makes arguments of logs negative
$\log_7\left(\frac{1}{16}+3\right)$ and $\log_7\left(\frac{1}{16}\right)$ are ok	No need to exclude from solution set
$x = \dfrac{1}{16}$	Final answer

Example 3: Solve

$\log x + \log(x-8) = \log 9$	Combine left side to a single log
$\log x(x-8) = \log 9$	Take both sides to power of 10 (base) to clear the logs
$x(x-8) = 9$	Distribute
$x^2 - 8x = 9$	Subtract 9 from both sides
$x^2 - 8x - 9 = 0$	Factor
$(x-9)(x+1) = 0$	Set each factor equal to zero
$x - 9 = 0$ or $x + 1 = 0$	Solve both equations
$x = 9, -1$	Check if this makes arguments of logs negative
$\log 9$ and $\log(9-8)$ are ok $\log -1$ and $\log(-1-8)$ are undefined	Exclude -1 from solution
$x = 9$	Final answer

Example 4: Solve

$\ln(2-x)+\ln(5-x)=\ln 18$	Combine left side to a single log
$\ln(2-x)(5-x)=\ln 18$	Take both sides to power of e (base) to clear the logs
$(2-x)(5-x)=18$	FOIL left side
$10-7x+x^2=18$	Subtract 18 and reorder terms
$x^2-7x-8=0$	Factor
$(x-8)(x+1)=0$	Set each factor equal to zero
$x-8=0$ or $x+1=0$	Solve both equations
$x=8,-1$	Check if this makes arguments of logs negative
$\ln(2-8)$ and $\ln(5-8)$ are undefined $\ln(2-(-1))$ and $\ln(5-(-1))$ are ok	Exclude 8 from solution
$x=-1$	Final answer

Example 5: Solve

$[\log_3 x]^2-5\log_3 x+6=0$	Notice quadratic "shape", factor
$(\log_3 x-3)(\log_3 x-2)=0$	Set each factor equal to zero
$\log_3 x-3=0$ or $\log_3 x-2=0$	Solve each equation for the log
$\log_3 x=3$ or $\log_3 x=2$	Convert to exponential equations
$3^3=x$ or $3^2=x$	Evaluate the exponents
$x=27,9$	Check if this makes arguments of logs negative
$\log_3 27$ and $\log_3 9$ are ok	No need to exclude from solution set
$x=27,9$	Final answer

Example 6: Solve

$\log_2 x + \log_x 16 = 4$	Need a common base, so we change the base to 2
$\log_2 x + \dfrac{\log_2 16}{\log_2 x} = 4$	Clear the fraction by multiplying by $\log_2 x$
$[\log_2 x]^2 + \log_2 16 = 4\log_2 x$	Simplify $\log_2 16$, "2 to what power is 16?"
$[\log_2 x]^2 + 4 = 4\log_2 x$	Notice quadratic "shape", make equation equal zero
$[\log_2 x]^2 - 4\log_2 x + 4 = 0$	Factor
$(\log_2 x - 2)^2 = 0$	Square root both sides
$\log_2 x - 2 = 0$	Add 2 to both sides
$\log_2 x = 2$	Convert to exponential equation
$2^2 = x$	Simplify
$4 = x$	Check if this makes arguments of logs negative
$\log_2 4$ is ok	No need to exclude from solution set
$x = 4$	Final answer

Example 7: Solve

$\log_{49} x + \log_{\sqrt{7}} x = 5$	Change to base of $\log_7$ as 49 and $\sqrt{7}$ are powers of 7
$\dfrac{\log_7 x}{\log_7 49} + \dfrac{\log_7 x}{\log_7 \sqrt{7}} = 5$	Simplify denominators
$\dfrac{\log_7 x}{2} + \dfrac{\log_7 x}{\frac{1}{2}} = 5$	Multiply numerators by reciprocals
$\dfrac{1}{2}\log_7 x + 2\log_7 x = 5$	Combine like terms ($\log_7 x$)
$\dfrac{5}{2}\log_7 x = 5$	Multiply both sides by reciprocal
$\log_7 x = 2$	Convert to an exponential equation
$7^2 = x$	Simplify
$49 = x$	Check if this makes arguments of logs negative
$\log_{49} 49$ and $\log_{\sqrt{7}} 49$ are ok	No need to exclude from solution set
$x = 49$	Final answer

Example 8: Solve

$10\log_4 x - 3\log_8 x = 4$	Change to base of $\log_2$ as 4 and 8 are powers of 2
$10 \cdot \frac{\log_2 x}{\log_2 4} - 3 \cdot \frac{\log_2 x}{\log_2 8} = 4$	Simplify denominators
$10 \cdot \frac{\log_2 x}{2} - 3 \cdot \frac{\log_2 x}{3} = 4$	Divide out denominators with numerators
$5\log_2 x - \log_2 x = 4$	Combine like terms
$4\log_2 x = 4$	Divide both sides by 4
$\log_2 x = 1$	Convert to an exponential equation
$2^1 = x$	Simplify
$2 = x$	Check if this makes arguments of logs negative
$\log_4 2$ and $\log_8 2$ are ok	No need to exclude from solution set
$x = 2$	Final answer

4.4 Solving Equations with Logarithms Practice

Solve each of the following equations for x

1. $\log x + \log(x+9) = 1$
2. $\log x - \log(x+3) = 1$
3. $\log(x+9) - \log x = 1$
4. $\log(2x+1) - \log(x-9) = 1$
5. $\log_4(x+3) + \log_4(x-3) = 2$
6. $\log_8(x+1) - \log_8 x = \log_8 4$
7. $\log x^2 = (\log x)^2$
8. $(\log_3 x)^2 - \log_3(x^2) = 3$
9. $\log \sqrt{x} = \sqrt{\log x}$
10. $\log(\log x) = 2$
11. $\log_5 \sqrt{x^2+1} = 1$
12. $\log \sqrt[3]{x^2} + \log \sqrt[3]{x^4} = \log(2^{-3})$
13. $\log_6 x + \log_6(x+1) = 1$
14. $\log_9(x+1) = \frac{1}{2} + \log_9 x$
15. $\log_2(x+4) = 2 - \log_2(x+1)$
16. $\log(2x+4) + \log(x-2) = 1$
17. $\ln x + \ln(x+1) = \ln 2$
18. $\log(x+3) - \log(x-2) = 2$
19. $\log_2(2x^2+4) = 5$
20. $\log(x-6) + \log(x+3) = 1$
21. $\log x - \log 5 = \log 2 - \log(x-3)$
22. $(\ln x)^3 = \ln(x^4)$
23. $\log(6x+5) - \log 3 = \log 2 - \log x$
24. $(\log x)^3 = \log(x^4)$
25. $\log(x^2) - \log(x-1) = 1$
26. $(\log x)^2 - 3\log(x) + 2 = 0$
27. $\frac{1}{3}\log 27 + \log(9-3) = \log x$
28. $3 + \log_2(3) + \log_2(x) = \log_2(96)$
29. $\log(2x) - 2\log x = -1$
30. $\frac{1}{\log(x)-1} = \frac{2}{\log(x)+1}$
31. $\log_2(3-x) + \log_2(1-x) = 3$
32. $\log_3(2-x) + \log_3(4-x) = 1$
33. $\log_5(2-x) + \log_5(6-x) = 1$
34. $\log_2(3-x) + \log_2(7-x) = 5$
35. $\log_3(5-x) + \log_3(3-x) = 1$
36. $\log_2(3-x) + \log_2(6-x) = 2$
37. $\log_2(7-x) + \log_2(5-x) = 3$
38. $\log_6(7-x) + \log_6(1-x) = 3$

39. $\log_4(4-x)+\log_4(1-x)=1$

40. $\log_3 x+\log_9 x+\log_{27} x=5.5$

41. $\log(x-4)+\log(x+3)=\log(5x+4)$

42. $\ln(x^2+1)-\frac{1}{2}\ln(x^2-2x+1)=\ln 5$

43. $2\log_3(x-2)+\log_3(x-4)^2=0$

44. $\log_2\left(\frac{x-2}{x-1}\right)=\log_2\left(\frac{3x-7}{3x-1}\right)$

45. $2\log_2\left(\frac{x-7}{x-1}\right)+\log_2\left(\frac{x-1}{x+1}\right)=1$

46. $\log(10x^2)\cdot\log(x)=1$

47. $2\log_9 x+9\log_x 3=10$

48. $\log_x(125x)\cdot(\log_{25} x)^2=1$

49. $\log_2 x+2\log_x 8=5$

50. $\log_3 x+2\log_x 9=5$

51. $\log_3 x-\log_x 27=2$

52. $\log_4 x+4\log_x 64=8$

53. $\log_{\sqrt{2}} x+3\log_x 4=8$

54. $\log_{\sqrt{3}} x+4\log_x 27=14$

55. $\log(\log x)+\log(\log x^3-2)=0$

56. $x^2\log_x(27)\cdot\log_9(x)=x+4$

57. $\log_x(2)-\log_4(x)+\frac{7}{6}=0$

58. $x^{\log(x)}=100x$

59. $x^{\log(x)}=\frac{x^3}{100}$

60. $(x+1)^{\log(x+1)}=100(x+1)$

61. $x^{\log(x)}=3x$

62. $2^{\frac{3}{\log_3(x)}}=\frac{1}{64}$

63. Solve for t: $P=P_0e^{kt}$

64. Solve for t: $T=T_0+(T_1-T_2)e^{-kt}$

65. Solve for n: $PV^n=c$

66. Solve for Q: $\log_a Q=\frac{1}{3}\log_a y+b$

67. Solve for x: $\log_a x=b+\log_a b$

68. Solve for y: $\log_a y=\log_a b-a$

69. Solve for P: $\log_b P=b-\log_b(aP)$

70. $\log(x+1)=\log(2x^2+3)-\log(2x-5)-1$

71. $\left[\log_2\left(\frac{1+x}{1-x}\right)\right]^2-5\log_2\left(\frac{1+x}{1-x}\right)+4=0$

72. $[\log(2x-1)]^2-3\log(2x-1)-10=0$

73. $\log(x-3)+\log(x+6)=\log(2)+\log(5)$

74. $\log_5(x-2)+2\log_5(x^3-2)+\log_5(x-2)^{-1}=4$

75. $\log_2(x+2)^2+\log_2(x+10)^2=4\log_2(3)$

76. $\log_3(5x-2)-2\log_3\sqrt{3x+1}=1-\log_3(4)$

77. $\log(3x-2)-2=\dfrac{1}{2}\log(x+2)-\log(50)$

78. $\log_{3x+7}(9+12x+4x^2)=4-\log_{2x+3}(6x^2+23x+21)$

79. $\log_{0.5x}x^2-14\log_{16x}x^3+40\log_{4x}\sqrt{x}=0$

4.5 Applications of Logarithms and Exponents

Basic Exponential Equation

$$f(t) = Pe^{kt}$$

Where k is the growth rate (if positive) or decay rate (if negative), t is time and P is the principle or starting amount.

Half-Life

Half-life is the time it takes for a substance to decay to 50% of its initial mass. Using $P = 1$, half this would be $f(t) = 0.5$. Solving the exponential equation for t and k we get the following formulas:

$$t = \frac{\ln 0.5}{k}$$

$$k = \frac{\ln 0.5}{t}$$

Example 1: Lead 185, ^{185}Pb, has a half-life of 6.3 seconds. How much of a 1000 g mass will remain in one minute?

First we will need to know the decay constant, k

$k = \frac{\ln 0.5}{t}$	The half-life is 6.3 seconds. Use this for t
$k = \frac{\ln 0.5}{6.3}$	Evaluate on calculator
$k \approx -0.11$	Using basic exponential equation with $P = 1000$, $t = 60$
$f(60) = 1000e^{-0.11 \cdot 60}$	Evaluate on calculator
$f(60) = 1.3585$ g	Final answer

Doubling Time

The time it takes for a growing amount to double is the doubling time. Use $P = 1$, double this would be $f(t) = 2$. Solving the exponential equation for t and k we get the following formulas:

$$t = \frac{\ln 2}{k}$$

$$k = \frac{\ln 2}{t}$$

Example 2: The world currently has a population growth rate of 1.14%. At this rate, how long will it take the Earth's population to double? (assume t is in years)

We first need to know the constant, k, which is given as a percent which must be convert to a decimal

$k = 0.0114$	Substitute into double time equation
$t = \dfrac{\ln 2}{0.0114}$	Evaluate on calculator
$t \approx 60.8024$ years	Final answer

General Growth/Decay Equation

The above formulas can be generalized in the following way: The time it takes for a substance to grow/decay to a times its original size can be found with the general growth/decay equation. Using $P = 1$, this would give $f(t) = a$. Solving the exponential equation for t and k we get the following formulas:

$$t = \frac{\ln a}{k}$$

$$k = \frac{\ln a}{t}$$

Continuous Compound Interest

If one deposits a principal P at an interest rate r, then after t years we can represent the resultant amount A as

$$A = Pe^{rt}$$

Example 3: How long does it take an amount of money to triple if invested at 1% compounded continuously?

Notice that this problem does not specify the principal amount. We will call it P. Because we want to know how much time will pass for this principal to triple, let $A = 3P$. Convert the interest rate into a decimal, $r = 0.01$,

$3P = Pe^{0.01t}$ Divide both sides by P

$3 = e^{0.01t}$ Convert to natural log

$\ln 3 = 0.01t$ Divide by 0.01

$\frac{\ln 3}{0.01} = t$ Evaluate on calculator

$t \approx 109.861$ years Final answer

Logistic Growth Equation

A growth rate of k and a maximum sustainable population M, grows in such a way that the population over time t is given by Q. B, which are an equation constant that must be solved for, given an initial population and a later population where:

$$Q(t) = \frac{M}{1 + Be^{kt}}$$

Example 4: A population of wolves consists of 1000 wolves. The area will sustain 2000 wolves. After 4 years the population is 1200. How long will it take for the population to reach 1500 wolves?

$$Q(0) = 1000$$

$$1000 = \frac{2000}{1 + Be^{k0}} \to 1000 = \frac{2000}{1 + B}$$

$1000(1 + B) = 2000$ Multiplying each side by (1 + B)

1 + B = 2 Dividing each side by 1000

B = 1 Subtracting each side by 1

$1200 = \frac{2000}{1+1e^{4k}}$ Substitute value for B into equation

$1200(1 + e^{4k}) = 2000$ Multiplying each side by $(1 + e^{4k})$

$1 + e^{4k} = \frac{5}{3}$ Dividing both sides by 1200

$e^{4k} = \frac{2}{3}$ Subtracting both sides by 1

$\ln e^{4k} = \ln\frac{2}{3}$ Multiply each side by ln

$4k = \ln\frac{2}{3}$ Divide each side by 4

k = -0.101366

Now that we know what B and k are, we can input our information into the equation again and solve for t.

$1500 = \frac{2000}{1+e^{-0.101366t}}$ Our equation

$1 + e^{-0.101366t} = \frac{5}{3}$ Skipping ahead (refer to prior steps when solving for k)

$t = (\ln\frac{1}{3})/-0.101366$ Last step

t = 10.84 years Final answer

Newton's Law of Cooling

If an object at temperature T_0 is surrounded by air of temperature T_a, it will gradually cool in such a way that T is the approximate resulting temperature over time t, where

$$T = T_a + (T_0 - T_a)e^{kt}$$

Example 5: You have some water boiling at a temperature of 212ºF. IF you leave it in a room with a temperature of 75ºF, and $k = -0.09$, find the temperature after an hour.

In this case, $T_0 = 212$ and $T_a = 75$. We also have $t = 60$ (60 min = 1 hr) and we are given $k = -0.09$. Substituting into our equation gives:

$T = 75 + (212 - 75)e^{(-0.09)(60)}$ Evaluate on calculator

$T = 75.6188$ºF Final answer

Example 6: You pull a whole chicken out of an oven. The chicken has a temperature of 165ºF. If $k = -0.016$ when time is in minutes, and the room has a temperature of 70ºF, how long will it take before the chicken cools to 120ºF?

Here, $T = 120$, $T_0 = 165$, $T_a = 70$ and we are given $k = -0.016$

$120 = 70 + (165 - 70)e^{(-0.016)t}$ Simplify parentheses

$120 = 70 + 95e^{-0.016t}$	Subtract 70 from both sides
$50 = 95e^{-0.016t}$	Divide by 95
$\frac{50}{95} = e^{-0.016t}$	Convert to natural log
$\ln\left(\frac{50}{95}\right) = -0.016t$	Divide both sides by -0.016
$\frac{\ln\left(\frac{50}{95}\right)}{-0.016} = t$	Evaluate on calculator
$t \approx 40.1159$ minutes	Final answer

Frequency of Keys on a Piano

The frequency of the nth key on a standard piano (with the first key, low A having a frequency measuring 27.5 Hz) is given by the function $f(n) = 27.5\left(\sqrt[12]{2}\right)^{n-1}$.

Example 7: Middle C has a frequency of about 261.626 Hz. Where does it lie on the keyboard?

The number 261.626 is the frequency and is substituted for $f(n)$

$261.626 = 27.5\left(\sqrt[12]{2}\right)^{n-1}$	Divide both sides by 27.5
$\frac{261.626}{27.5} = \left(\sqrt[12]{2}\right)^{n-1}$	Take a natural log of both sides
$\ln\left(\frac{261.626}{27.5}\right) = \ln\left(\sqrt[12]{2}\right)^{n-1}$	Move exponent to front
$\ln\left(\frac{261.626}{27.5}\right) = (n-1)\ln\sqrt[12]{2}$	Divide by the natural log

$$\frac{\ln\left(\frac{261.626}{27.5}\right)}{\ln \sqrt[12]{2}} = n - 1$$ Add 1 to both sides

$$\frac{\ln\left(\frac{261.626}{27.5}\right)}{\ln \sqrt[12]{2}} + 1 = n$$ Evaluate on calculator

$n \approx 40^{\text{th}}$ key Final answer

4.5 Applications of Logarithms and Exponents Practice

1. The half-life of ^{234}U is 2.52×10^5 years. How much of a 100 gram sample remains after 10,000 years?

2. How much of a 100 gram specimen of ^{22}Na remains after 7 years if its half-life is 2.6 years?

3. ^{242}Cm has a half-life of 163 days. How much remains of 10 grams after one week?

4. ^{239}Np has a half-life of 2.237 days. How much remains of 10 grams after one week?

5. How much of 10 grams of ^{189}Pb remains after a day if its half-life is 4.98 hours?

6. How much of 25 grams of $^{234}Pu234$ remains after a day if its half-life is 4.98 hours?

7. What is the half-life of cesium 137 (in years) if the decay constant is $k = -0.0231$?

8. What is the half-life of strontium 90 (in years) if the decay constant is $k = -0.0246$?

9. What is the half-life of krypton (in years) if the decay constant is $k = -0.0641$?

10. If the population of Anchorage, Alaska, continued to grow at its 1970 - 1980 rate, the city would double in size approximately every 5.4 years. Estimate its 1990 population if it was 48,081 in 1970.

11. Aurora, Colorado, would double in size every 8 years if the population continued to grow it its 1970 - 1975 rate. Estimate its 1985 population if the population was 74,974 in 1970.

12. Every 36 years, Little Rock, Arkansas would double in population if the population continued to grow at its 1960-1980 rate. Estimate the 1985 population of Little Rock if it was 107,813 in 1960.

13. Springfield, Missouri, had a population of 95,865 in 1960, and grew from 1960 to 1980 at a rate that would cause it to double every 42.23 years. Estimate Springfield's population in 1990.

14. The population of the state of Texas grew from 1950 to 1980 at an annual rate of approximately 2%. If the population in 1950 was 7,711,194, what was the population in 1980?
15. Estimate the population of Texas in 1990, using the information in problem 14.

16. Florida grew in population between 1940 and 1980 at an annual rate of 4.09%. If the population was 1,897,414 in 1940, what was the population in 1980?

17. What is the anticipated population of Florida in the year 2000 if the data in problem 16 remains constant?

18. The population of Los Angeles was 1,970,358 in 1950. It has grown since at an annual rate of 1.36%. Estimate its population in the years 1980, 1990, and 2000.

19. San Jose, CA, has had a phenomenal 6% annual growth since 1950. Estimate its population in the years 1980, 1990, and 2000, if its population was 95,280 in 1950.

20. The decay constant of Strontium-90 is –.0248. What amount of 250 mg of strontium-90 is present after 5 years?

21. Radium has a decay constant of –.0004. How much of 1000 mg of radium remains after a century?

22. The growth rate of a certain cell culture is proportional to its size. Initially, 2×10^5 cells were present. In 10 hours there were approximately 8×10^5 cells. How long will it take until there are 10^6 cells present.

23. The decay constant for cobalt 60 is $k = -0.13$ when time is measured in years. Find the half-life of cobalt 60.

24. Radioactive potassium is also used for dating fossils. It has a half-life of 1.3 billion years. Determine the decay constant.

25. The size of a certain insect population is given by $P = 300e^{0.1t}$ where t is measured in days. After how many days will the population equal 600?, 1200?

26. The half-life of carbon 14 is approximately 5590 years. Find the decay constant of carbon 14.

27. Some bone artifacts were found at the Lindenmeier site in Northeastern Colorado and tested for their carbon 14 content. If 25% of the original carbon 14 was still present, what is the probable age of the artifacts?

28. An artifact was discovered at the Debert site in Nova Scotia. Tests showed that 28% of the original carbon 14 was still present. What is the probable age of the artifact?

29. An artifact was found and tested for its carbon 14 content. If 12% of the original carbon 14 was still present, what is the probable age?

30. An artifact was found and tested for its carbon 14 content. If 85% of the original carbon 14 was still present, what is the probable age?

31. Sandals woven from strands of tree bark were found in Fort Rock Cave in Oregon. The bark has a carbon 14 ratio of 0.34 times the ratio found in living bark. Estimate the age of the sandals.

32. A 4500 year old wooden chest was found in the tomb of the twenty-fifth century B.C. Chaldean king Meskalamdug of Ur. What carbon 14 ratio would you expect to find in the wooden chest?

33. Prehistoric cave paintings were discovered in the Lascaux cave in France. Charcoal from the site was found to have a carbon 14 ratio of 15%. Estimate the age of the paintings.

34. Before radiocarbon dating was used, historians estimated that the age of the tomb of Vizier Hemaka, in Egypt, was constructed about 4900 years ago. After radiocarbon dating became available, wood samples from the tomb were analyzed and it was determined that the carbon 14 ratio was about 51%. Estimate the age of the tomb on this basis.

35. Analyses of the oldest campsites of ancient man in the Western Hemisphere reveal a carbon 14 ratio of 22.6%. Determine the probable age of the campsites.

36. The Dead Sea Scrolls are a collection of ancient manuscripts discovered in caves along the west bank of the Dead Sea. (The discovery occurred by accident when an Arab herdsman of the Taamireh tribe was searching for a stray goat.) When the linen wrappings on the scrolls were analyzed, the carbon 14 ratio was found to be 72.3%. Estimate the age of the scrolls using this information.

37. An island in the Pacific Ocean is contaminated by fallout from a nuclear explosion. If the strontium 90 is 100 times the level that scientists believe is "safe," how many years will it take for the island to once again be "safe" for human habitation? The half-life of strontium 90 is 28 years.

38. If a bacteria culture doubles in size every 20 minutes, how long will it take for a population of 10^4 to grow to 10^8 bacteria?

39. A certain cell culture grows at a rate proportional to the size of the culture. During a 10 hour experiment the culture doubled in size every three hours. At the end of the experiment approximately 10^5 cells were present. How many cells were present at the beginning of the experiment?

40. By 1974 the United States had an estimated 80 million gallons of radioactive products form nuclear power plants and other nuclear reactors. These waste products were stored in various sorts of containers (made of such materials as stainless steel and cement), and the containers were buried in the ground and the ocean. Scientists feel that the waste products must be prevented from contaminating the rest of the earth until more than 99.99% of the radioactivity is gone (that is, until the level is less than .0001 times the original level). If a storage cylinder contains waste products whose half-life is 1500 years, how many years must the container survive without leaking? (Note: Some of the containers are already leaking.)

41. The atmospheric pressure P (in psi) is approximated by

$$P = 14.7e^{-0.1h}$$

where h is the altitude above sea level in miles.

   a. Mt. McKinley, in Alaska, is the highest point in North America. The elevation is 20,320 feet. What is the pressure at its summit? (1 mile = 5280 ft.)

   b. The lowest land point in the world is the Dead Sea (Isreal-Jordan), where the elevation is 1299 feet below sea level. What is the atmospheric pressure at this point?

42. A healing law for skin wounds states that

$$A = A_0e^{-0.1t}$$

where A is the number of square centimeters of unhealed skin after t days when the original area of the wound was A. How many days does it take for half of the wound to heal?

43. A law of light absorption of a medium for a beam of light passing through is given by

$$I = I_0e^{-rt}$$

where I_0 is the original intensity of the beam in lumens, and I is the intensity after passing through t cm of a medium whose absorption coefficient is r. Find the intensity of a 100 lumen beam after it passes through 2.54 cm of a medium with absorption coefficient of .095.

44. A learning curve describes the rate at which a person learns certain specific tasks. If N is the number of words per minute typed by a student, then

$$N = 80(1 - e^{-0.16t})$$

where t is the number of days of instruction. Assuming Joe is an average student, what is his typing rate after 20 days of instruction?

45. Members of a discussion group tend to be ranked exponentially by the number of times they participate in a discussion. For a group of ten, the number of times, the nth ranked participant, takes part is given by
$$P_n = P_1 e^{0.11(1-n)}$$
where P_1 is the number of times the first-ranked person participates in the discussion. For each 100 times the top-ranked participant enters the discussion, how many times should the bottom-ranked person be expected to participate?

If an object at temperature T_0 is surrounded by air at a temperature T_a, it will gradually cool so that the temperature T is given by
$$T = T_a + (T_0 - T_a)e^{kt}$$
where the constant k depends upon the particular object being measured and t is given in appropriate time units (minutes or hours, etc.). This formula is called Newton's law of cooling.

46. Solve the formula for the constant k.

47. You draw a tub of hot water ($k = -0.09$ for time measured in minutes) for a bath. The water is 100°F when drawn and the room is 72°F. If you are called away to the phone, what is the temperature of the water 20 minutes later when you get in?

48. You take a batch of chocolate chip cookies from the oven (250°F) when the room temperature is 74°F. If the cookies cool for 20 minutes and $k = -0.095$ when time is measured in minutes, what is the temperature of the cookies?

49. It is known that the temperature of a given object fell from 120°F to 70°F in an hour when placed in 20°F air. What was the temperature of the object after 30 minutes?

50. An object is initially 100°F. In air of 40°F it cools to 45°F in 20 minutes.
    a. What is its temperature in 30 minutes?
    b. How long will it take the object to cool to 40°F?
    c. How long will it take this object to cool to 75°F?

The logistics equation $Q(t) = \frac{M}{1+be^{kt}}$ is a growth equation used to describe how a population quantity Q grows over time t. In this equation M represents the maximum population that a particular environment will support.

51. A population of 200 birds is introduced into an environment which will support 800 birds. After 6 weeks the population of birds has grown to 450. How long will it take to reach a population of 600 birds?

52. A population of Northern White Bears consists of 60 bears. Population growth is encouraged and after 3 years the population expands to 75 bears. If the environment will only support 1200 bears, how long will it take the population to reach 1000 bears?

53. An artificial laboratory environment will support 1,000,000 dung flies. At the beginning of an experiment 200 flies are introduced into the environment. After 30 minutes the number of flies has increased to 6000 flies. How long will it take fly population to increase to 800,000 flies?

54. The forest service has determined that Black Lake is capable of supporting a population of 8000 fish and that the population needs to be at least 6000 fish before the lake can be opened up for fishing. At the beginning of a survey period it is estimated that there are 800 fish in the lake. Two years later the population grows to 1275 fish. How long will it take until the lake can be opened up for fishing?

55. A desert environment is capable of supporting a population of 15000 rock lizards. Environmentalists surveying the desert estimate a population of 1875 lizards. Five years later a second survey estimates a population of 9800 lizards. How long will it take until the population is estimated to be 12000 lizards?

56. An environment the will support 300 lemurs is observed to have 50 lemurs in residence. Three years later the population grows to 80 lemurs. How many years will it take until the population reaches 130 lemurs?

57. The jungles of Soporphia have enough water to support a population of 5000 white tailed lynx. A survey team investigating the mating habits of the lynx find that the population is 200 lynx. Ten years later the population has grown to 900 lynx. How long will it take the population to reach 2500 lynx?

58. A genetics experimenter creates an artificial environment that will support 103500 blood flies. He puts 2300 of the flies in the environment and observes their growth patterns. After 25 weeks the population of flies grows to 53000 flies. How long will it take the population to grow to 85000 flies?

59. The mountains of Kalahlalawai have enough food and water to support a population of 6080 llama. A National Geographic documentary team counts 380 llama in the mountains at the beginning of their project. 5 years later the population has grown to 1145 llama. How many years will it be until the population of llama reaches 2200?

60. The Siberian wasteland is capable of supporting 4384 snow lizards. An initial survey determines that the population is 137 lizards. Six years later the population reaches 260 snow lizards. Howl long until the population of snow lizards reaches 2100?

pH is a measure of the acidity of an aqueous solution. If. H_3O+ refers to the molarity (concentration of hydronium ions in moles per liter) of the solution,

$$pH = -\log(H_3O+)$$

61. Pure water lies at 7 on the pH scale. Find its molarity

62. Unripe orange juice lies at 2.9 on the pH scale. Find its molarity

63. Household bleach lies at 12.6 on the pH scale. Find its molarity

Sound Pressure Level: The sound level L_p, in decibels, of a sound with pressure given by p_{rms} (root mean square) in micropascals against a reference sound of p_{ref} is given by

$$L_p = 20\log\left(\frac{p_{rms}}{p_{ref}}\right)$$

We usually set $p_{ref} = 20$ micropascals (the lower threshold of human hearing) as the standard, whereas underwater we set $p_{ref} = 1$ micropascals.

64. Even with hearing protection, short term exposure to sounds louder than 140 dB can cause permanent damage. Find the sound pressure of this noise in micropascals.

65. A power drill has a sound pressure of about 2700 micropascals. Find the decibel level of this power drill.

66. You invest $5000 at 8% compounded continuously. How much remains after five years?

67. An 8.5% account earns continuous interest. If $2500 is deposited for 5 years, what is the total accumulated?

68. You invest $12123 at 1% compounded continuously. How much remains after six months?

69. You lend $100 at 10% continuous interest. If you are repaid 2 months later, what is owed?

70. If $1000 is invested at 16% compounded continuously, how long will it take to quadruple?

71. How long does it take an amount of money to double if invested at 2% compounded continuously?

72. How long does it take an amount of money to triple if invested at 28% compounded continuously?

The frequency of the nth key on the piano is given by the function $f(n) = 27.5\left(\sqrt[12]{2}\right)^{n-1}$.

73. Find the frequency of the 88^{th} key on the piano

74. The A key above middle C has a standard frequency of 440 Hz. Where does it lie on the piano?

75. Find the ratio of frequencies of two notes that lie an octave apart, or 12 keys apart.

76. Find the ratio of frequencies of a perfect fifth, or exactly 7 keys apart.

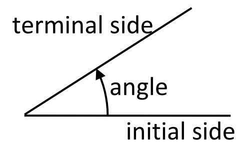
Chapter 5

Trigonometry

5.1 Angles

Because many applications involving circles also involve a rotation of the circle, it is natural to introduce a measure for the rotation, or angle, between two rays (line segments) emanating from the center of a circle. The angle measurement you are most likely familiar with is degrees, so we'll begin there.

The **measure of an angle** is a measurement between two intersecting lines, line segments or rays, starting at the **initial side** and ending at the **terminal side**. It is a rotational measure not a linear measure.



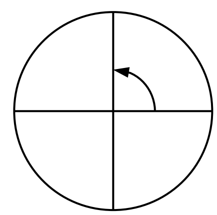
Measuring Angles

A **degree** is a measurement of angle. One full rotation around the circle is equal to 360 degrees, so one degree is $\frac{1}{360}$ of a circle.

An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees is written 90º.

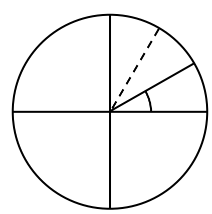
When measuring angles on a circle, unless otherwise directed, we measure angles in **standard position**: starting at the positive horizontal axis and with counter-clockwise rotation.

Example 1: Give the degree measure of the angle shown on the circle.



The vertical and horizontal lines divide the circle into quarters. Since one full rotation is 360 degrees or 360º, each quarter rotation is $\frac{360°}{4} = 90°$ or 90 degrees.

Example 2: Show an angle of 30º on the circle.



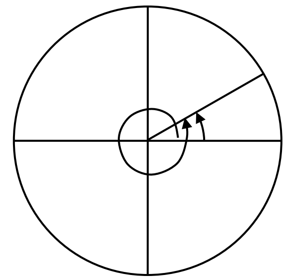
An angle of is $\frac{1}{3}$ of 90º, so by dividing a quarter rotation into thirds, we can sketch a line at 30º.

Going Greek

When representing angles using variables, it is traditional to use Greek letters. Here is a list of commonly encountered Greek letters.

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Working with Angles in Degrees



Notice that since there are 360 degrees in one rotation, an angle greater than 360 degrees would indicate more than 1 full rotation. Shown on a circle, the resulting direction in which this angle's terminal side points would be the same as for another angle between 0 and 360 degrees. These angles would be called **coterminal**.

Coterminal Angles

After completing their full rotation based on the given angle, two angles are **coterminal** if they terminate in the same position, so their terminal sides coincide (point in the same direction). Adding or subtracting a full rotation, 360 degrees, would result in an angle with terminal side pointing in the same direction; we can find coterminal angles by adding or subtracting 360 degrees.

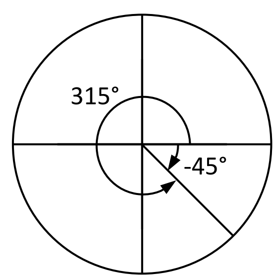
Example 3: Find an angle θ that is coterminal with 800º, where $0° \leq \theta < 360°$

$800°$	Subtract 360º
$800° - 360° = 440°$	Subtract 360º
$440° - 360° = 80°$	Final answer

By finding the coterminal angle between 0 and 360 degrees, it can be easier to see which direction the terminal side of an angle points in.

On a number line a positive number is measured to the right and a negative number is measured in the opposite direction (to the left). Similarly a positive angle is measured counterclockwise and a negative angle is measured in the opposite direction (clockwise).

Example 4: Show the angle $-45°$ on the circle and find a positive angle α that is coterminal and $0° \leq \alpha < 360°$.

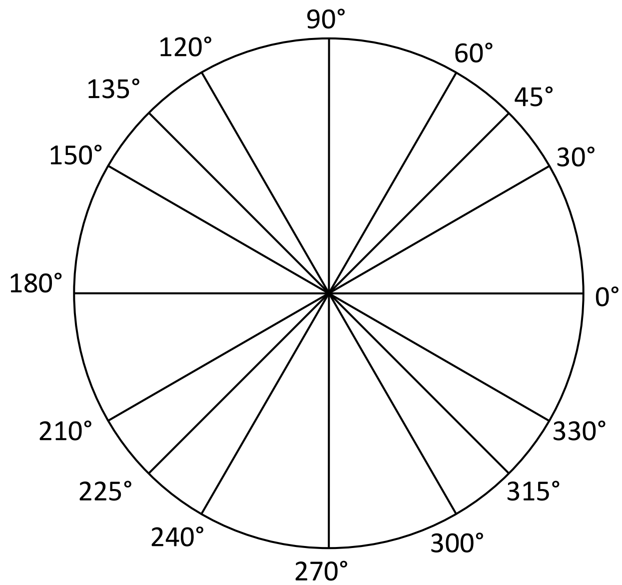


Since 45 degrees is half of 90 degrees, we can start at the positive horizontal axis and measure clockwise half of a 90 degree angle.

Since we can find coterminal angles by adding or subtracting a full rotation of 360 degrees, we can find a positive coterminal angle here by adding 360 degrees:

$$-45° + 360° = 315°$$

It can be helpful to have a familiarity with the frequently encountered angles in one rotation of a circle. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown to the right. Memorizing these angles and understanding their properties will be very useful as we study the properties associated with angles

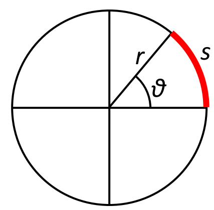


Angles in Radians

While measuring angles in degrees may be familiar, doing so often complicates matters since the units of measure can get in the way of calculations. For this reason, another measure of angles is commonly used. This measure is based on the distance around a circle.

Arclength

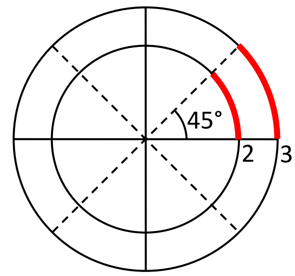
Arclength is the length of an arc, s, along a circle of radius r subtended (drawn out) by an angle θ. It is the portion of the circumference between the initial and terminal sides of the angle.



The length of the arc around an entire circle is called the circumference of a circle. The circumference of a circle is $C = 2\pi r$. The ratio of the circumference to the radius, produces the constant 2π. Regardless of the radius, this ratio is always the same, just as how the degree measure of an angle is independent of the radius.

To elaborate on this idea, consider two circles, one with radius 2 and one with radius 3. Recall the circumference (perimeter) of a circle is $C = 2\pi r$, where r is the radius of the circle. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$.

Drawing a 45 degree angle on the two circles, we might be interested in the length of the arc of the circle that the angle indicates.



In both cases, the 45 degree angle draws out an arc that is $\frac{1}{8}$ of the full circumference, so for the smaller circle, the arclength $\frac{1}{8}(4\pi) = \frac{1}{2}\pi$, and for the larger circle, the length of the arc or arclength $\frac{1}{8}(6\pi) = \frac{3}{4}\pi$.

Notice what happens if we find the *ratio* of the arclength divided by the radius of the circle:

Smaller circle: $\frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi$

Larger circle: $\frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$

The ratio is the same regardless of the radius of the circle – it only depends on the angle. This property allows us to define a measure of the angle based on arclength.

Radians

The **radian measure** of an angle is the ratio of the length of the circular arc subtended by the angle to the radius of the circle.

In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then radian measure is found by $\theta = \frac{s}{r}$

If the circle has radius 1, then the radian measure corresponds to the length of the arc.

Because radian measure is the ratio of two lengths, it is a **unitless measure**. It is not necessary to write the label "radians" after a radian measure, and if you see an angle that is not labeled with "degrees" or the degree symbol, you should assume that it is a radian measure.

Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360º. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$1 \text{ rotation} = 360° = 2\pi \text{ radians}$$

$$\frac{1}{2} \text{ rotation} = 180° = \pi \text{ radians}$$

$$\frac{1}{4} \text{ rotation} = 90° = \frac{\pi}{2} \text{ radians}$$

Example 1: Find the radian measure of one third of a full rotation.

$\frac{1}{3}$ rotation — The arclength is one third of the circumference

$C = \frac{1}{3}(2\pi r) = \frac{2\pi r}{3}$ — radian measure is the arclength divided by the radius

$\frac{2\pi r}{3} \cdot \frac{1}{r} = \frac{2\pi}{3}$ — Final answer

Converting Between Radians and Degrees

1 degree is $\frac{\pi}{180}$ radians, or, to convert from degrees to radians, multiply by $\frac{\pi \text{ radians}}{180°}$

1 radian is $\frac{180}{\pi}$ degrees, or, to convert from radians to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$

Example 2: Convert $\frac{\pi}{6}$ radians to degrees.

$\frac{\pi}{6}$ — Multiply by $\frac{180°}{\pi}$

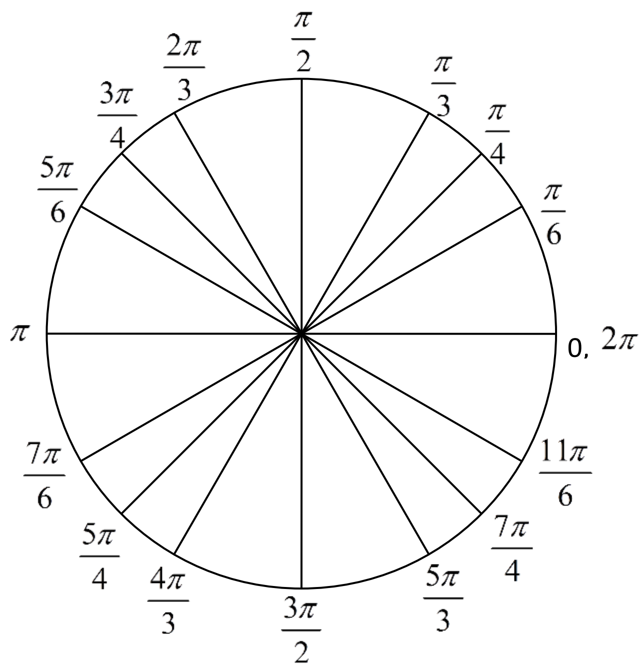
$\frac{\pi}{6} \cdot \frac{180°}{\pi} = 30°$ — Final answer

Example 3: Convert 15 degrees to radians.

$15°$ — Multiply by $\frac{\pi}{180°}$

$15° \cdot \frac{\pi}{180°} = \frac{\pi}{12}$ — Final answer

Just as we listed all the common angles in degrees on a circle, we should also list the corresponding radian values for the common measures of a circle corresponding to degree multiples of 30, 45, 60, and 90 degrees. As with the degree measurements, it would be advisable to commit these to memory.



We can work with the radian measures of an angle the same way we work with degrees.

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians.

Example 4: Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$

$$\frac{19\pi}{4} \qquad \text{Subtract } 2\pi$$

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} = \frac{11\pi}{4} \qquad \text{Subtract } 2\pi$$

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4} \qquad \text{Final answer}$$

Arclength

Recall that the radian measure of an angle was defined as the ratio of the arclength of a circular arc to the radius of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arclength along a circle given an angle.

Arclength on a Circle

The length of an arc, s, along a circle of radius r subtended by angle θ in radians is $s = r\theta$

Example 5: Mercury orbits the sun at a distance of approximately 36 million miles. In one Earth day, it completes 0.0114 rotation around the sun. If the orbit was perfectly circular, what distance through space would Mercury travel in one Earth day?

0.0114 rotation	Convert to radians, multiply by 2π
$2\pi(0.0114) = 0.0716$ radians	Use formula $s = r\theta$
$S = 36(0.0716) = 2.578$ million miles	Final answer

Linear and Angular Velocity

When your car drives down a road, it makes sense to describe its speed in terms of miles per hour or meters per second. These are measures of speed along a line, also called linear velocity. When a point on a circle rotates, we would describe its angular velocity, or rotational speed, in radians per second, rotations per minute, or degrees per hour.

Angular and Linear Velocity

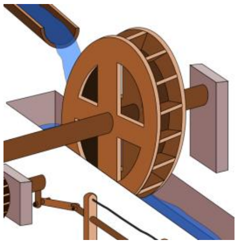
As a point moves along a circle of radius r, its **angular velocity,** ω, can be found as the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t}$$

The **linear velocity**, $\boldsymbol{v}$, of the point can be found as the distance travelled, arclength s, per unit time, t.

$$v = \frac{s}{t}$$

Example 7: A water wheel completes 1 rotation every 5 seconds. Find the angular velocity in radians per second.



1 rotation (2π) in 5 seconds	Use formula for angular velocity, $\omega = \frac{\theta}{t}$
$\omega = \frac{2\pi}{5} = 1.257$ rad/sec	Final answer

Combining the definitions above with the arclength equation, $s = r\theta$, we can find a relationship between angular and linear velocities. The angular velocity equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arclength equation gives $s = r\theta = r\omega t$.

Substituting this into the linear velocity equation gives

$$v = \frac{s}{t} = \frac{r\omega t}{t} = r\omega$$

Relationship Between Linear and Angular Velocity

When the angular velocity is measured in radians per unit time, linear velocity and angular velocity are related by the equation

$$v = r\omega$$

Example 8: A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is travelling down the road.

Note: Here we have an angular velocity and need to find the corresponding linear velocity, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

180 rotations/minute	Convert 1 rotation is 2π radians
$180 \cdot 2\pi = 360\pi$ radians/minute	Using new linear velocity formula, $v = r\omega$
$v = (14)(360\pi) = 5040\pi$ in/min	Optional: convert to miles per hour
$\frac{5040\pi \text{ in}}{\text{min}} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} = 14.99\, mph$	Final answer

Changing to Degree, Minutes, Seconds

Recall: $1° = 60'$ (one degree equals sixty minutes)

$1' = 60"$ (one minute equals sixty seconds)

The notation $\theta = 73°\ 56'\ 18"$ refers to an angle θ that measures 73 degrees, 56 minutes, 18 seconds.

Example 9: If $\theta = 3$, approximate θ in terms of degrees, minutes, and seconds.

3	Multiply by $\frac{180}{\pi}$
$3\left(\frac{180}{\pi}\right) = 171.8873$	Multiply decimal portion by $60'$
$171° + 0.8873(60') = 171° + 53.238'$	Multiply decimal portion by 60”
$171° + 53' + 0.238(60)$=171° 53' 14"	Final answer

Example 10: Express $19°\ 47'\ 23"$ as a decimal, to the nearest ten-thousandth of a degree

Note: Since $1' = \left(\frac{1}{60}\right)$ degrees and $1" = \left(\frac{1}{60}\right)'$ we also have $1" = \left(\frac{1}{3600}\right)$ degrees

$19° + 47' + 23"$	Multiply minutes and degrees by above conversion
$19 + 47\left(\frac{1}{60}\right) + 23\left(\frac{1}{3600}\right)$	Multiply
$19 + 0.7833 + 0.0064$	Add
$19.7897°$	Final answer

5.1 Angles Practice

1. Indicate each angle on a circle: $30°$, $300°$, $-135°$, $70°$, $\frac{2\pi}{3}$, $\frac{7\pi}{4}$

2. Indicate each angle on a circle: $25°$, $315°$, $-115°$, $80°$, $\frac{7\pi}{6}$, $\frac{3\pi}{4}$

3. Convert the angle $180°$ to radians.

4. Convert the angle $30°$ to radians.

5. Convert the angle $\frac{5\pi}{6}$ from radians to degrees.

6. Convert the angle $\frac{11\pi}{6}$ from radians to degrees.

7. Find the angle between $0°$ and $360°$ that is coterminal with a $685°$ angle.

8. Find the angle between $0°$ and $360°$ that is coterminal with a $451°$ angle.

9. Find the angle between $0°$ and $360°$ that is coterminal with a $-1746°$ angle.

10. Find the angle between $0°$ and $360°$ that is coterminal with a $-1400°$ angle.

11. Find the angle between 0 and 2π in radians that is coterminal with the angle $\frac{26\pi}{9}$.

12. Find the angle between 0 and 2π in radians that is coterminal with the angle $\frac{17\pi}{3}$.

13. Find the angle between 0 and 2π in radians that is coterminal with the angle $-\frac{3\pi}{2}$.

14. Find the angle between 0 and 2π in radians that is coterminal with the angle $-\frac{7\pi}{6}$.

15. On a circle of radius 7 miles, find the length of the arc that subtends a central angle of 5 radians.

16. On a circle of radius 6 feet, find the length of the arc that subtends a central angle of 1 radian.

17. On a circle of radius 12 cm, find the length of the arc that subtends a central angle of 120 degrees.

18. On a circle of radius 9 miles, find the length of the arc that subtends a central angle of 800 degrees.

19. Find the distance along an arc on the surface of the Earth that subtends a central angle of 5 minutes (1 minute = $\frac{1}{60}$ degree). The radius of the Earth is 3960 miles.

20. Find the distance along an arc on the surface of the Earth that subtends a central angle of 7 minutes (1 minute = $\frac{1}{60}$ degree). The radius of the Earth is 3960 miles.

21. On a circle of radius 6 feet, what angle in degrees would subtend an arc of length 3 feet?

22. On a circle of radius 5 feet, what angle in degrees would subtend an arc of length 2 feet?

23. A truck with 32-in.-diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

24. A bicycle with 24-in.-diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

25. A wheel of radius 8 in. is rotating 15°/sec. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/sec?

26. A wheel of radius 14 in. is rotating 0.5 rad/sec. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/sec?

27. A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

28. When being burned in a writable CD-R drive, the angular speed of a CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed.

29. You are standing on the equator of the Earth (radius 3960 miles). What is your linear and angular speed?

30. The restaurant in the Space Needle in Seattle rotates at the rate of one revolution per hour. [UW]

a. Through how many radians does it turn in 100 minutes?

b. How long does it take the restaurant to rotate through 4 radians?

c. How far does a person sitting by the window move in 100 minutes if the radius of the restaurant is 21 meters?

31. Express θ in terms of degrees, minutes, and seconds, to the nearest second.

a. $\theta = 2$

b. $\theta = 5$

c. $\theta = 4$

32. Express the angle in terms of degrees, minutes, and seconds, to the nearest second.

a. 63.169°

b. 310.6215°

c. 81.7238°

5.2 Right Triangle Trigonometry

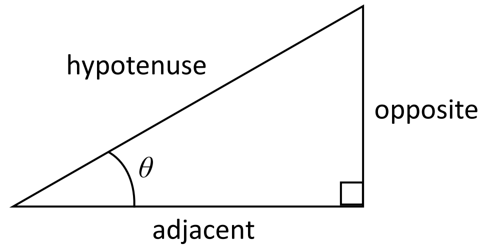
Right Triangle Relationships

Given a right triangle with an angle of θ

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{adjacent}$$

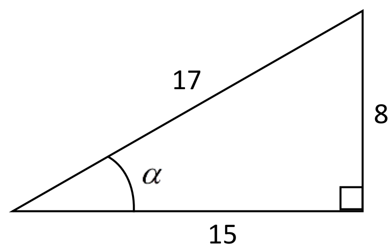


A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

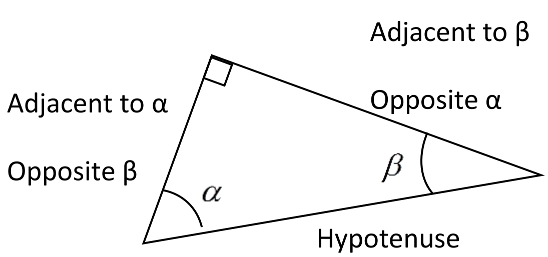
Example 1: Given the triangle shown, find the value for $\cos(\alpha)$.

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$$



When working with general right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the sine and cosine of either of the two acute angles in the triangle.



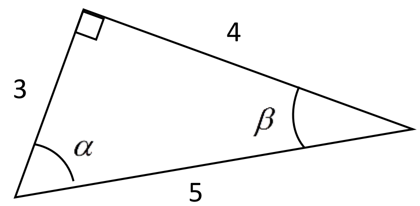
Example 2: Using the triangle shown, evaluate $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\beta)$, $\sin(\beta)$

$$\cos(\alpha) = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$

$$\sin(\alpha) = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos(\beta) = \frac{\text{adjacent to } \beta}{\text{hypotenuse}} = \frac{4}{5}$$

$$\sin(\beta) = \frac{\text{opposite of } \beta}{\text{hypotenuse}} = \frac{3}{5}$$



You may have noticed that in the above example that $\cos(\alpha) = \sin(\beta)$ and $\cos(\beta) = \sin(\alpha)$. This makes sense since the side opposite α is also adjacent to β. Since the three angles in a triangle need to add to π, or 180 degrees, then the other two angles must add to $\frac{\pi}{2}$, or 90 degrees, so $\beta = \frac{\pi}{2} - \alpha$, and $\alpha = \frac{\pi}{2} - \beta$. Since $\cos(\alpha) = \sin(\beta)$, then $\cos(\alpha) = \sin\left(\frac{\pi}{2} - \alpha\right)$.

Cofunction Identities

The **cofunction identities** for sine and cosine

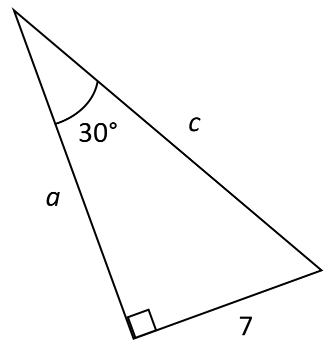
$$\cos(\theta) = sin\left(\frac{\pi}{2} - \theta\right)$$

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

In the previous examples we evaluated the sine and cosine on triangles where we knew all three sides of the triangle. Right triangle trigonometry becomes powerful when we start looking at triangles in which we know an angle but don't know all the sides.

Example 3: Find the unknown sides of the triangle pictured here.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$ Substitute using $\theta = 30$



$$\sin(30) = \frac{7}{c}$$ Solve for c

$$c = \frac{7}{\sin(30)}$$ Use calculator

$$c = 14$$ Pythagorean Theorem

$$a^2 + (7)^2 = (14)^2$$ Square

$$a^2 + 49 = 196$$ Subtract 49

$$a^2 = 147$$ Take square root

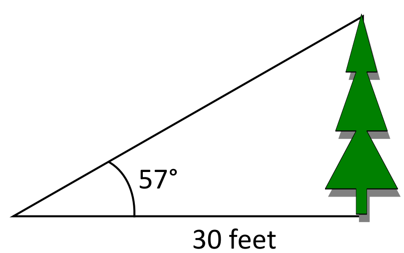
$$a = \sqrt{147}$$ Simplify: $\sqrt{147} = \sqrt{7^2 \cdot 3} = 7\sqrt{3}$

$$a = 7\sqrt{3}, c = 14$$ Final answer

Notice that if we know at least one of the non-right angles of a right triangle and one side, we can find the rest of the sides and angles.

Example 4: To find the height of a tree, a person walks to a point 30 feet from the base of the tree, and measures the angle from the ground to the top of the tree to be 57 degrees. Find the height of the tree.

We can introduce a variable, h, to represent the height of the tree. The two sides of the triangle that are most important to us are the side opposite the angle, the height of the tree we are looking for, and the adjacent side, the side we are told is 30 feet long.



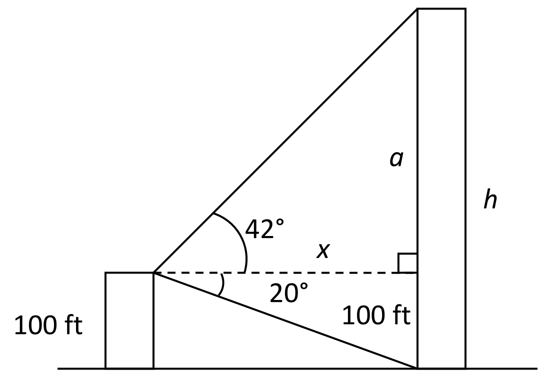
The trigonometric function which relates the side opposite of the angle and the side adjacent to the angle is the tangent.

$$\tan(57) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{30}$$ Solve for h

$$h = 30\tan(57)$$ Solve with a calculator

$$h = 46.2 \text{ ft}$$ Final answer

Example 5: A person standing on the roof of a 100 foot building is looking towards a skyscraper a few blocks away, wondering how tall it is. She measures the angle of declination from the roof of the building to the base of the skyscraper to be 20 degrees and the angle of inclination to the top of the skyscraper to be 42 degrees.



To approach this problem, it would be good to start with a picture. Although we are interested in the height, h, of the skyscraper, it can be helpful to also label other unknown quantities in the picture – in this case the horizontal distance x between the buildings and a, the height of the skyscraper above the person.

To start solving this problem, notice we have two right triangles. In the top triangle, we know one angle is 42 degrees, but we don't know any of the sides of the triangle, so we don't yet know enough to work with this triangle.

In the lower right triangle, we know one angle is 20 degrees, and we know the vertical height measurement of 100 ft. Since we know these two pieces of information, we can solve for the unknown distance x.

$$\tan(20) = \frac{\text{opposite}}{\text{adjacent}} = \frac{100}{x}$$ Solve for x

$$x = \frac{100}{\tan(20)}$$ Now that we have found x, we solve the top right triangle

$$\tan(42) = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{x} = \frac{a}{\frac{100}{\tan(20)}}$$ Multiply by reciprocal

$$\tan(42) = \frac{a\tan(20)}{100}$$ Solve for a, multiply by $\frac{100}{\tan(20)}$

$$\frac{100\tan(42)}{\tan(20)} = a$$ Use a calculator

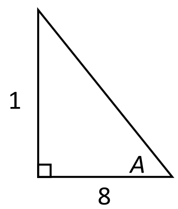
$a = 247.4$

247.4 + 100 = 347.4 ft Final answer

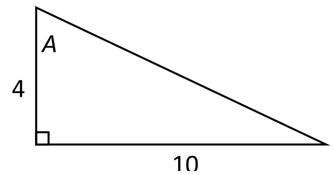
5.2 Right Triangle Trigonometry Practice

In each of the triangles below, find $\sin(A)$, $\cos(A)$, $\tan(A)$.

1.

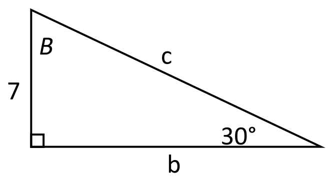


2.

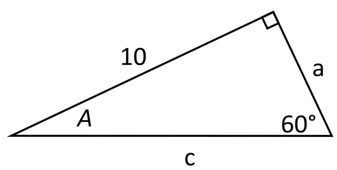


In each of the following triangles, solve for the unknown sides and angles.

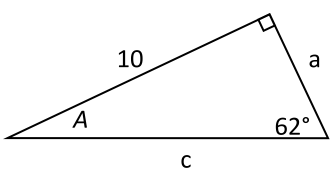
3.



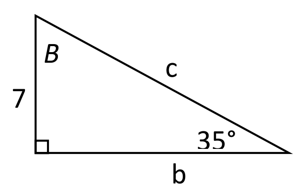
4.



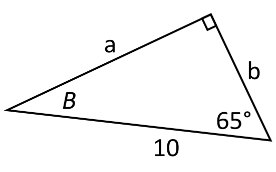
5.



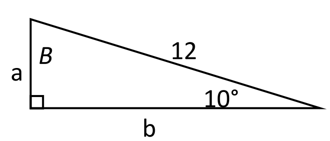
6.



7.



8.



9. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

10. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

11. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

12. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

13. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36° and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

14. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43° and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

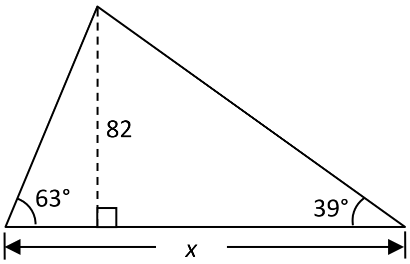
15. A 200 foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15° and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

16. A 400 foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18° and that the angle of depression to the bottom of the tower is 3°. How far is the person from the monument?

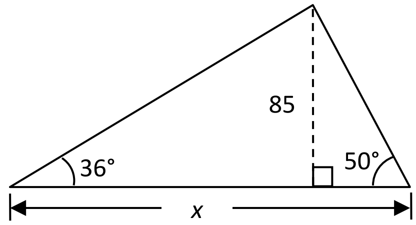
17. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

18. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

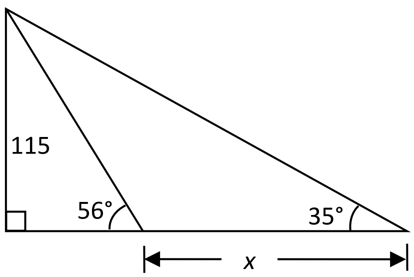
19. Find the length *x*.



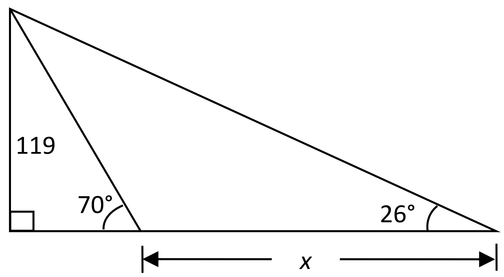
20. Find the length *x*.



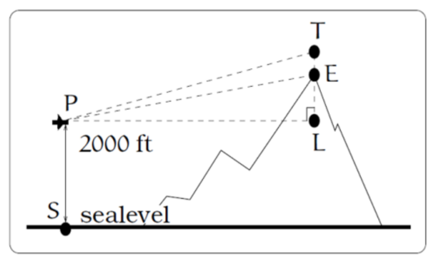
21. Find the length *x*.



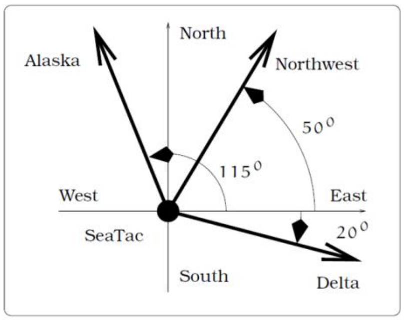
22. Find the length *x*.



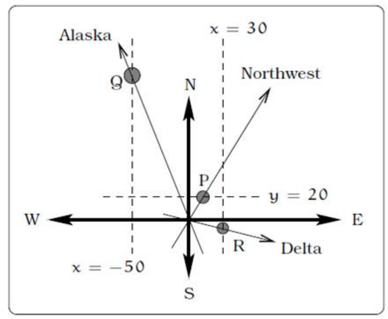
23. A plane is flying 2000 feet above sea level toward a mountain. The pilot observes the top of the mountain to be 18^o above the horizontal, then immediately flies the plane at an angle of 20^o above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain? [UW]



24. Three airplanes depart SeaTac Airport. A Northwest flight is heading in a direction 50° counterclockwise from east, an Alaska flight is heading 115° counterclockwise from east and a Delta flight is heading 20° clockwise from east. Find the location of the Northwest flight when it is 20 miles north of SeaTac. Find the location of the Alaska flight when it is 50 miles west of SeaTac. Find the location of the Delta flight when it is 30 miles east of SeaTac. (please note in diagram b, x = 20, and y = 30 [UW]

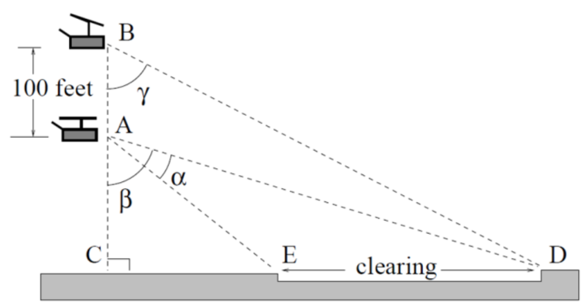


(a) The flight paths of three airplanes.

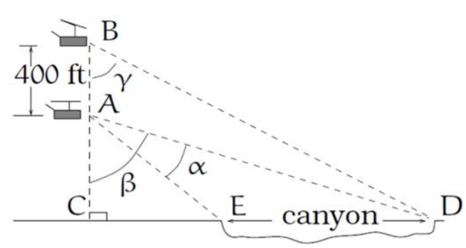


(b) Modeling the paths of each flight.

25. The crew of a helicopter needs to land temporarily in a forest and spot a flat piece of ground (a clearing in the forest) as a potential landing site, but are uncertain whether it is wide enough. They make two measurements from A (see picture) finding α = 25° and β = 54°. They rise vertically 100 feet to B and measure γ = 47°. Determine the width of the clearing to the nearest foot. [UW]



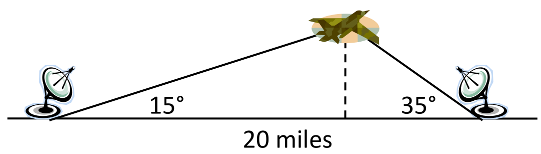
26. A Forest Service helicopter needs to determine the width of a deep canyon. While hovering, they measure the angle γ = 48° at position B (see picture), then descend 400 feet to position A and make two measurements: α = 13° (the measure of ∠EAD), β = 53° (the measure of ∠CAD). Determine the width of the canyon to the nearest foot. [UW]



5.3 Non-Right Triangles: Laws of Sines and Cosines

Although right triangles allow us to solve many applications, it is more common to find scenarios where the triangle we are interested in does not have a right angle.

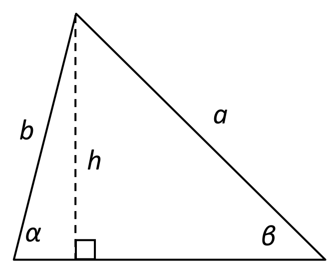
Two radar stations located 20 miles apart both detect a UFO located between them. The angle of elevation measured by the first station is 35 degrees. The angle of elevation measured by the second station is 15 degrees. What is the altitude of the UFO?



We see that the triangle formed by the UFO and the two stations is not a right triangle. Of course, in any triangle we could draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles, but it would be nice to have methods for working directly with non-right triangles. In this section we will expand upon the right triangle trigonometry we learned in 5.2, and adapt it to non-right triangles.

Law of Sines

Given an arbitrary non-right triangle, we can drop an altitude, which we temporarily label h, to create two right triangles.



Using the right triangle relationships,

$\sin(\alpha) = \frac{h}{b}$ and $\sin(\beta) = \frac{h}{a}$

Solving both equations for h, we get $b\sin(\alpha) = h$ and $a\sin(\beta) = h$. Since the h is the same in both equations, we establish $b\sin(\alpha) = a\sin(\beta)$. Dividing, we conclude that

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b}$$

Had we drawn the altitude to be perpendicular to side b or a, we could similarly establish

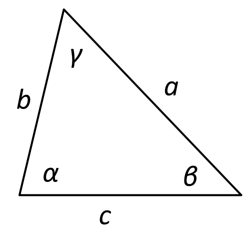
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c} \quad \text{and} \quad \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Collectively, these relationships are called the **Law of Sines**.

Law of Sines

Given a triangle with angles and sides opposite labeled as shown, the ratio of sine of angle to length of the opposite side will always be equal, or, symbolically,

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$



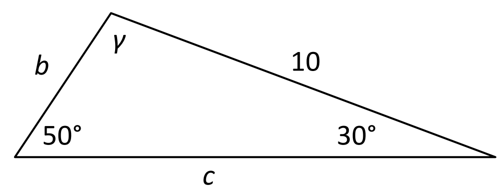
For clarity, we call side a the corresponding side of angle α.

Similarly, we call angle α, the corresponding angle of side a.

Likewise for side b and angle β, and for side c and angle γ.

When we use the law of sines, we use any pair of ratios as an equation. In the most straightforward case, we know two angles and one of the corresponding sides.

Example 1: In the triangle shown here, solve for the unknown sides and angle.

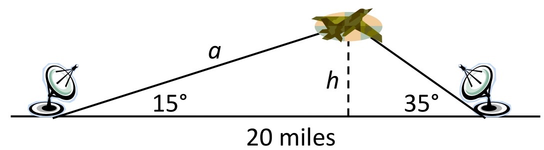


Solving for the unknown angle is relatively easy, since the three angles must add to 180 degrees.

γ	Subtract each angle from 180
$\gamma = 180 - 50 - 30 = 100$	Identify known angle and corresponding side
50° and 10	To find side b use its corresponding angle, 30º
$\frac{\sin(50)}{10} = \frac{\sin(30)}{b}$	Multiply both sides by b
$\frac{b\sin(50)}{10} = \sin(30)$	Solve for b, multiply by $\frac{10}{\sin(50)}$
$b = \frac{10\sin(30)}{\sin(50)}$	Use a calculator
$b = 6.527$	Similarly for c
$\frac{\sin(50)}{10} = \frac{\sin(100)}{c}$	Solve and use a calculator
$c = \frac{10\sin(100)}{\sin(50)} = 12.856$	Final answer

Example 2: Find the elevation of the UFO from the beginning of the section.

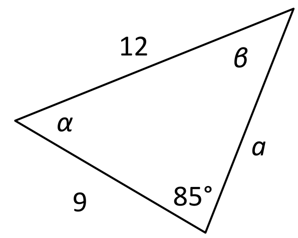
To find the elevation of the UFO, we first find the distance from one station to the UFO, such as the side a in the picture, then use right triangle relationships to find the height of the UFO, h.



γ	Subtract each angle from 180
$180 - 15 - 35 = 130$	Using 20 miles and 130º we set up Law of Sines
$\dfrac{\sin(130)}{20} = \dfrac{\sin(35)}{a}$	Solve for a, use a calculator
$a = \dfrac{20\sin(35)}{\sin(130)} = 14.975$	Use the right triangle relationship to solve for h
$\sin(15) = \dfrac{h}{14.975}$	Multiply by 14.975
$h = 14.975\sin(15) = 3.876$ miles	Final answer

In addition to solving triangles in which two angles are known, the Law of Sines can be used to solve for an angle when two sides and one corresponding angle are known.

Example 3 In the triangle shown here, solve for the unknown sides and angles.



In choosing which pair of ratios from the Law of Sines to use, we always want to pick a pair where we know three of the four pieces of information in the equation. In this case, we know the angle 85° and its corresponding side, so we will use that ratio. Since our only other known information is the side with length 9, we will use that side and solve for its corresponding angle.

$\dfrac{\sin(85)}{12} = \dfrac{\sin(\beta)}{9}$	Solve for $\sin(\beta)$
$\dfrac{9\sin(85)}{12} = \sin(\beta)$	Use inverse sine to find first solution
$\beta = \sin^{-1}\left(\dfrac{9\sin(85)}{12}\right) = 48.3438°$	With inverse sine there are two solutions, subtract from 180 to get second possible
$\beta = 180 - 48.3438 = 131.6562$	Find α for each case, subtract angles from 180
$\alpha = 180 - 85 - 48.3438 = 46.6562$	Notice the second option is impossible
$\alpha = 180 - 85 - 131.6562 = -35.6562$	
$\alpha = 46.6562, \beta = 48.3438$	Using the Law of Sines we can find a
$\dfrac{\sin(85)}{12} = \dfrac{\sin(46.6562)}{a}$	Solve for a, use calculator
$a = \dfrac{12\sin(46.6562)}{\sin(85)} = 8.7603$	Final answer

Notice that in the problem above, when we use Law of Sines to solve for an unknown angle, there can be two possible solutions. This is called the **ambiguous case**, and can arise when we know two sides and a non-included angle. In the ambiguous case we may find that a particular set of given information can lead to 2, 1 or no solution at all. However, when an accurate picture of the triangle or suitable context is available, we can determine which angle is desired.

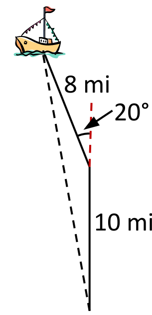
Example 4: Find all possible triangles if one side has length 4 opposite an angle of 50° and a second side has length 10.

$\frac{\sin(50)}{4}$	Using the side and angle we know, use Law of Sines
$\frac{\sin(50)}{4} = \frac{\sin(\alpha)}{10}$	Multiply by 10
$\frac{10\sin(50)}{4} = \sin(\alpha)$	Use the inverse sine
$\alpha = \sin^{-1}\left(\frac{10\sin(50)}{4}\right) = \text{undefined}$	Since the range of the sine function is [-1, 1], it is impossible for the sine value to be 1.915.
There are no triangles	Final answer

Example 5: Find all possible triangles if one side has length 6 opposite an angle of 50° and a second side has length 4.

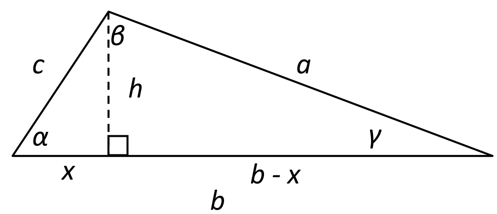
$\frac{\sin(50)}{6}$	Using given information set up Law of Sines
$\frac{\sin(50)}{6} = \frac{\sin(\alpha)}{4}$	Multiply by 4
$\frac{4\sin(50)}{6} = \sin(\alpha)$	Take the sine inverse
$\alpha = \sin^{-1}\left(\frac{4\sin(50)}{6}\right) = 30.71$	Subtract from 180 to find second case
$\alpha = 180 - 30.71 = 149.29$	Subtract all angles from 180 to find third angle
$\beta = 180 - 50 - 30.71 = 99.290$	Only the first option is possible.
$\beta = 180 - 50 - 149.29 = -19.29$	
$\alpha = 149.29°, \beta = 99.29°$	Using Law of Sines, find missing side
$\frac{\sin(50)}{6} = \frac{\sin(99.29)}{c}$	Solve for c, use a calculator
$c = \frac{6\sin(99.29)}{\sin(50)} = 7.73$	Final answer

Law of Cosines



Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat?

Unfortunately, while the Law of Sines lets us address many non-right triangle cases, it does not allow us to address triangles where the one known angle is included between two known sides, which means it is not a corresponding angle for a known side. For this, we need another tool.



Given an arbitrary non-right triangle, we can drop an altitude, which we temporarily label h, to create two right triangles. We will divide the base b into two pieces, one of which we will temporarily label x. From this picture, we can establish the right triangle relationship

$$\cos(\alpha) = \frac{x}{c} \quad \text{or, solving for } x, \qquad x = c\cos(\alpha)$$

Using the Pythagorean Theorem, we can establish

$$(b - x)^2 + h^2 = a^2 \quad \text{and} \quad x^2 + h^2 = c^2$$

Both of these equations can be solved for h^2

$$h^2 = a^2 - (b - x)^2 \quad \text{and} \quad h^2 = c^2 - x^2$$

Since the left side of each equation is h^2, the right sides must be equal

$$c^2 - x^2 = a^2 - (b - x)^2$$ Multiply out the right

$$c^2 - x^2 = a^2 - (b^2 - 2bx + x^2)$$ Distribute negative

$$c^2 - x^2 = a^2 - b^2 + 2bx - x^2$$ Add x^2 to both sides

$$c^2 = a^2 - b^2 + 2bx$$ Solve for a^2

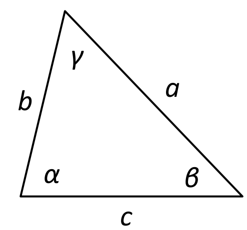
$$a^2 = b^2 + c^2 - 2bx$$ Substitute $x = c\cos(\alpha)$ from above

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$ Law of Cosines

This result is called the Law of Cosines. Depending upon which side we dropped the altitude down from, we could have established this relationship using any of the angles. The important thing to note is that the right side of the equation involves an angle and the sides adjacent to that angle – the left side of the equation involves the side opposite that angle.

Law of Cosines

Given a triangle with angles and opposite sides labeled as shown,



$$a^2 = c^2 + b^2 - 2bc\cos(\alpha)$$

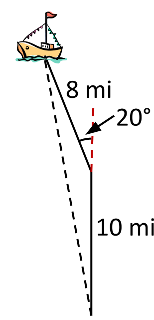
$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Notice that if one of the angles of the triangle is 90 degrees, $\cos(90) = 0$, so the formula simplifies to $c^2 = a^2 + b^2$

You should recognize this as the Pythagorean Theorem. Indeed, the Law of Cosines is sometimes called the **Generalized Pythagorean Theorem**, since it extends the Pythagorean Theorem to non-right triangles.

Example 6: Returning to our question from earlier, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat?

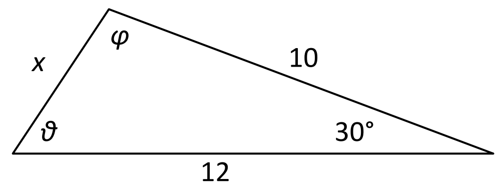


The boat turned 20 degrees, so the obtuse angle of the non-right triangle shown in the picture is the supplemental angle, $180 - 20 = 160°$.

With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle – the distance from the boat to port.

$x^2 = 8^2 + 10^2 - 2(8)(10)\cos(160)$	Evaluate right side on calculator
$x^2 = 314.3508$	Square root both sides
$x = 17.73$ miles	Final answer

Example 7: Find the unknown side and angles of this triangle.



Notice that we don't have both pieces of any side/angle pair, so the Law of Sines would not work with this triangle.

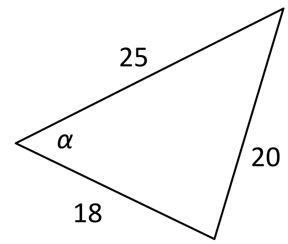
Since we have the angle included between the two known sides, we can turn to Law of Cosines.

$x^2 = 10^2 + 12^2 - 2(10)(12)\cos(30)$	Evaluate right side on calculator
$x^2 = 36.154$	Take the square root
$x = 6.013$	Now use law of sines to find another angle
$\dfrac{\sin(30)}{6.013} = \dfrac{\sin(\theta)}{10}$	Multiply by 10
$\dfrac{10\sin(30)}{6.013} = \sin(\theta)$	Inverse sine
$\theta = \sin^{-1}\left(\dfrac{10\sin(30)}{6.013}\right) = 56.256$	Subtract from 180 to get second case
$\theta = 180 - 56.256 = 123.744$	Find final angle for each case
Case 1: $\gamma = 180 - 30 - 56.256 = 93.744°$ Case 2: $\gamma = 180 - 30 - 123.744 = 26.256°$	Both cases work, we have two triangles
$x = 6.013, \theta = 56.256°, \gamma = 93.744°$ or $x = 6.013, \theta = 123.744°, \gamma = 26.256°$	Final answer

In addition to solving for the missing side opposite one known angle, the Law of Cosines allows us to find the angles of a triangle when we know all three sides.

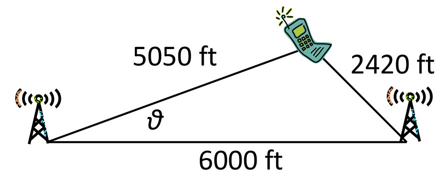
Example 8: Solve for the angle α in the triangle shown.

$20^2 = 18^2 + 25^2 - 2(18)(25)\cos(\alpha)$	Simplify
$400 = 949 - 900\cos(\alpha)$	Subtract 949
$-549 = -900\cos(\alpha)$	Divide -900
$\frac{549}{900} = \cos(\alpha)$	Inverse cosine
$\alpha = \cos^{-1}\left(\frac{549}{900}\right) = 52.410°$	Final answer



Notice that since the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases when using Law of Cosines to find an angle.

Example 9: On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is done by a process called triangulation, which works by using the distance from two known points. Suppose there are two cell phone towers within range of you, located 6000 feet apart along a straight highway that runs east to west, and you know you are north of the highway. Based on the signal delay, it can be determined you are 5050 feet from the first tower, and 2420 feet from the second. Determine your position north and east of the first tower, and determine how far you are from the highway.



For simplicity, we start by drawing a picture and labeling our given information. Using the Law of Cosines, we can solve for the angle θ.

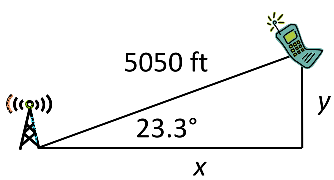
$$2420^2 = 6000^2 + 5050^2 - 2(5050)(6000)\cos(\theta)$$ Simplify

$$5856400 = 61501500 - 60600000\cos(\theta)$$ Subtract

$$-554646100 = -60600000\cos(\theta)$$ Divide

$$\frac{554646100}{60600000} = \cos(\theta)$$ Inverse cosine

$$\theta = \cos^{-1}\left(\frac{554646100}{60600000}\right) = 23.328°$$



Using this angle, use right triangles to find the position of the cell phone relative to the western tower.

$$\cos(23.3) = \frac{x}{5050}$$ Solve for x and y

$$\sin(23.3) = \frac{y}{5050}$$

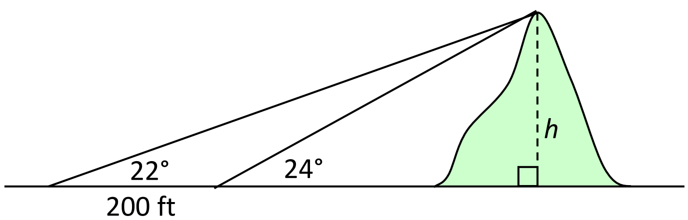
$$x = 5050\cos(23.3) = 4637.2 \text{ ft}$$
$$y = 5050\sin(23.3) = 1999.8 \text{ ft}$$ Final answer

You are about 4637 feet east and 2000 feet north of the first tower.

Note that if you didn't know whether you were north or south of the towers, our calculations would have given two possible locations, one north of the highway and one south. To resolve this ambiguity in real world situations, locating a position using triangulation requires a signal from a third tower.

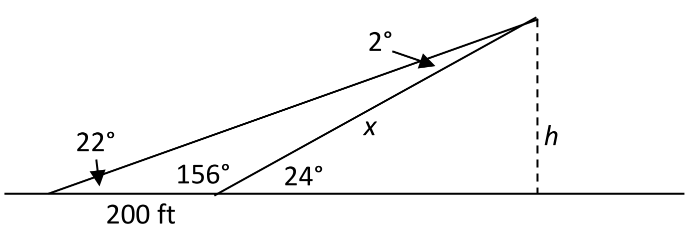
Example 10: To measure the height of a hill, a woman measures the angle of elevation to the top of the hill to be 24 degrees. She then moves back 200 feet and measures the angle of elevation to be 22 degrees. Find the height of the hill.

As with many problems of this nature, it will be helpful to draw a picture.



Notice there are three triangles formed here – the right triangle including the height h and the 22 degree angle, the right triangle including the height h and the 24 degree angle, and the (non-right) obtuse triangle including the 200 ft side. Since this is the triangle we have the most information for, we will begin with it. It may seem odd to work with this triangle since it does not include the desired side h, but we don't have enough information to work with either of the right triangles yet.

We can find the obtuse angle of the triangle, since it and the angle of 24 degrees complete a straight line – a 180 degree angle. The obtuse angle must be $180 - 24 = 156°$. From this, we can determine that the third angle is 2°. We know one side is 200 feet, and its corresponding angle is 2°, so by introducing a temporary variable x for one of the other sides (as shown below), we can use Law of Sines to solve for this length x



$$\frac{x}{\sin(22)} = \frac{200}{\sin(2)}$$ Solve for x

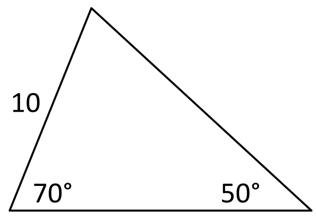
$$x = \frac{200\sin(22)}{\sin(2)} = 2146.77 \text{ ft}$$ Use right triangle properties

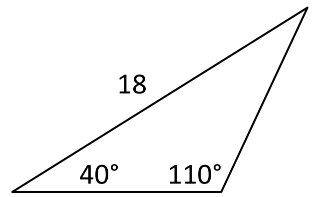
$$\sin(24) = \frac{h}{2146.77}$$ Solve for h

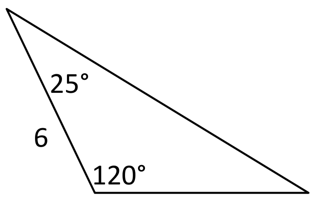
$$h = 2146.77\sin(24) = 873.17 \text{ ft}$$ Final answer

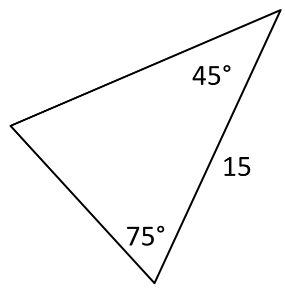
5.3 Non-Right Triangles: Laws of Sines and Cosines Practice

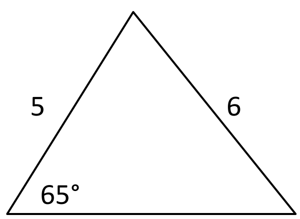
Solve for the unknown sides and angles of the triangles shown.

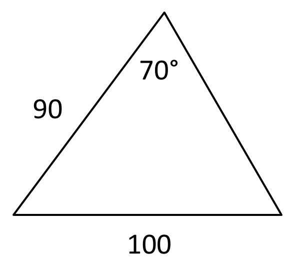
1. 

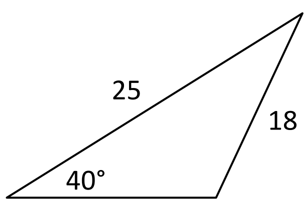
2. 

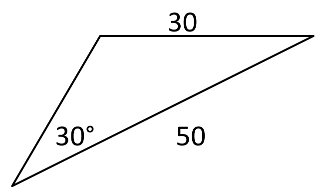
3. 

4. 

5. 

6. 

7. 

8. 

Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle for the unknown sides and angles if possible. If there is more than one solution, give both.

9. $\alpha = 43°, \gamma = 69°, b = 20$

10. $\alpha = 35°, \gamma = 73°, b = 19$

11. $\alpha = 119°, a = 26, b = 14$

12. $\gamma = 113°, b = 10, c = 32$

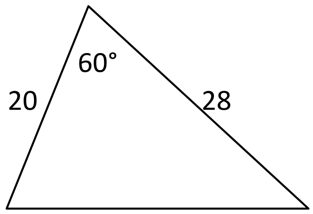
13. $\beta = 50°, a = 105, b = 45$

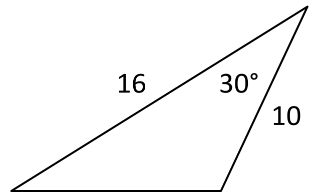
14. $\beta = 67°, a = 49, b = 38$

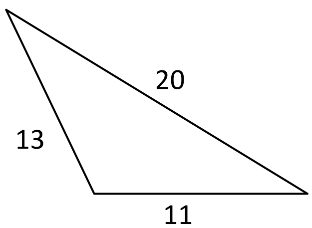
15. $\alpha = 43.1°, a = 184.2, b = 242.8$

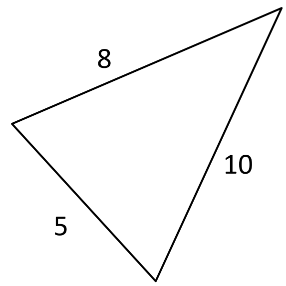
16. $\alpha = 36.6°, a = 186.2, b = 242.2$

Solve for the unknown sides and angles of the triangles shown.

17. 

18. 

19. 

20. 

Assume α is opposite side *a*, β is opposite side *b*, and γ is opposite side *c*. Solve each triangle for the unknown sides and angles if possible. If there is more than one possible solution, give both.

21. $\gamma = 41.2°, a = 2.49, b = 3.13$

22. $\beta = 58.7°, a = 10.6, c = 15.7$

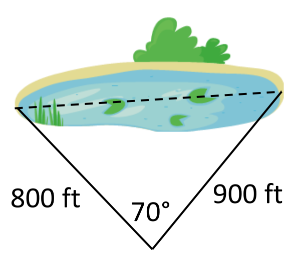
23. $\alpha = 120°, b = 6, c = 7$

24. $\gamma = 115°, a = 18, b = 23$

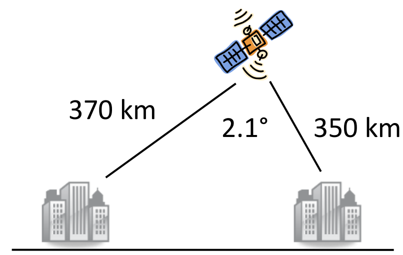
25. Find the area of a triangle with sides of length 18, 21, and 32.

26. Find the area of a triangle with sides of length 20, 26, and 37.

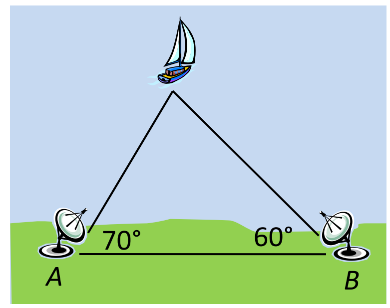
27. To find the distance across a small lake, a surveyor has taken the measurements shown. Find the distance across the lake.



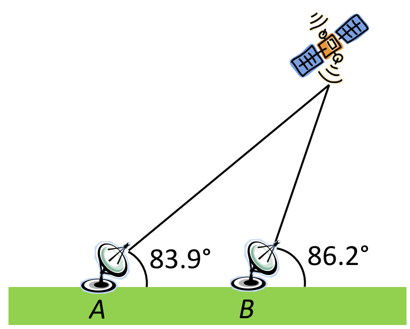
28. To find the distance between two cities, a satellite calculates the distances and angle shown (*not to scale*). Find the distance between the cities.



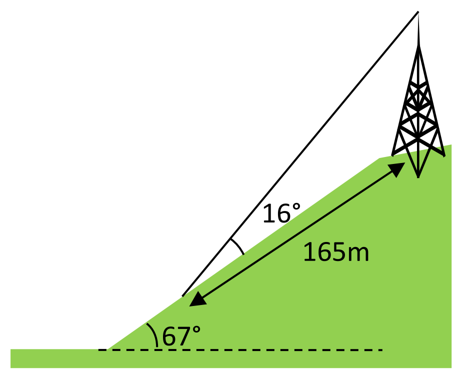
29. To determine how far a boat is from shore, two radar stations 500 feet apart determine the angles out to the boat, as shown. Find the distance of the boat from the station *A*, and the distance of the boat from shore.



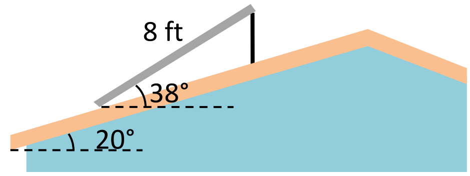
30. The path of a satellite orbiting the earth causes it to pass directly over two tracking stations *A* and *B*, which are 69 mi apart. When the satellite is on one side of the two stations, the angles of elevation at *A* and *B* are measured to be 83.9° and 86.2°, respectively. How far is the satellite from station *A* and how high is the satellite above the ground?



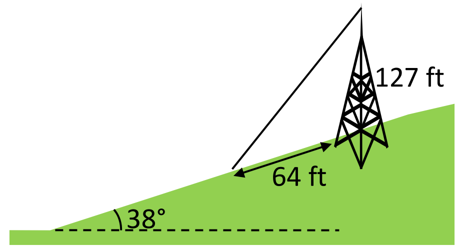
31. A communications tower is located at the top of a steep hill, as shown. The angle of inclination of the hill is 67°. A guy-wire is to be attached to the top of the tower and to the ground, 165 m downhill from the base of the tower. The angle formed by the guy-wire and the hill is 16°. Find the length of the cable required for the guy wire.



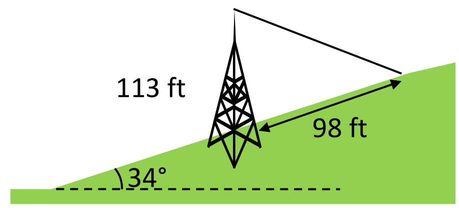
32. The roof of a house is at a 20° angle. An 8 foot solar panel is to be mounted on the roof, and should be angled 38° relative to the horizontal for optimal results. How long does the vertical support holding up the back of the panel need to be?



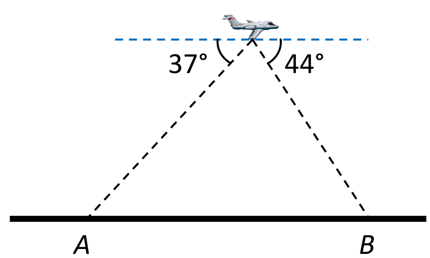
33. A 127 foot tower is located on a hill that is inclined 38° to the horizontal. A guy-wire is to be attached to the top of the tower and anchored at a point 64 feet downhill from the base of the tower. Find the length of wire needed.



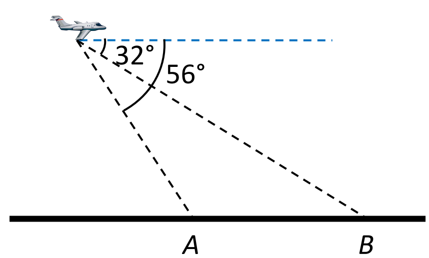
34. A 113 foot tower is located on a hill that is inclined 34° to the horizontal. A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.



35. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be 37° and 44°, as shown in the figure. Find the distance of the plane from point *A*, and the elevation of the plane.



36. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in the figure. Find the distance of the plane from point *A*, and the elevation of the plane.



37. To estimate the height of a building, two students find the angle of elevation from a point (at ground level) down the street from the building to the top of the building is 39°. From a point that is 300 feet closer to the building, the angle of elevation (at ground level) to the top of the building is 50°. If we assume that the street is level, use this information to estimate the height of the building.

38. To estimate the height of a building, two students find the angle of elevation from a point (at ground level) down the street from the building to the top of the building is 35°. From a point that is 300 feet closer to the building, the angle of elevation (at ground level) to the top of the building is 53°. If we assume that the street is level, use this information to estimate the height of the building.

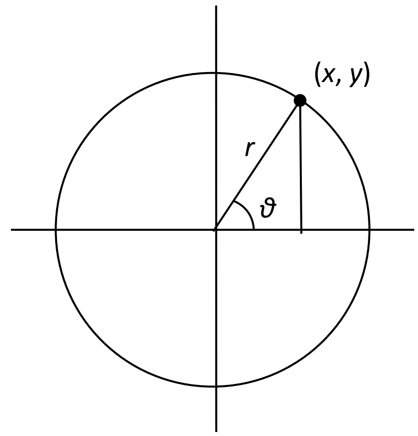
39. A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10 degrees to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

40. Two planes leave the same airport at the same time. One flies at 20 degrees east of north at 500 miles per hour. The second flies at 30 east of south at 600 miles per hour. How far apart are the planes after 2 hours?

5.4 Points on Circles

While it is convenient to describe the location of a point on a circle using an angle or a distance along the circle, relating this information to the x and y coordinates and the circle equation is an important application of trigonometry.

A distress signal is sent from a sailboat during a storm, but the transmission is unclear and the rescue boat sitting at the marina cannot determine the sailboat's location. Using high powered radar, they determine the distress signal is coming from a distance of 20 miles at an angle of 225 degrees from the marina. How many miles east/west and north/south of the rescue boat is the stranded sailboat?



In a general sense, to investigate this, we begin by drawing a circle centered at the origin with radius r, and marking the point on the circle indicated by some angle θ. This point has coordinates (x, y).

If we drop a line segment vertically down from this point to the x axis, we would form a right triangle inside of the circle.

No matter which quadrant our angle θ puts us in we can draw a triangle by dropping a perpendicular line segment to the x axis, keeping in mind that the values of x and y may be positive or negative, depending on the quadrant.

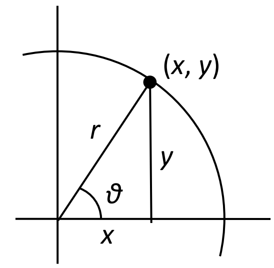
Additionally, if the angle θ puts us on an axis, we simply measure the radius as the x or y with the other value being 0, again ensuring we have appropriate signs on the coordinates based on the quadrant.

Triangles obtained from different radii will all be similar triangles, meaning corresponding sides scale proportionally. While the lengths of the sides may change, the *ratios* of the side lengths will always remain constant for any given angle.

To be able to refer to these ratios more easily, we will give them names. Since the ratios depend on the angle, we will write them as functions of the angle θ.

Sine and Cosine

For the point (x, y) on a circle of radius r at an angle of θ, we can define two important functions as the ratios of the sides of the corresponding triangle:



The **sine** function: $\sin(\theta) = \frac{y}{r}$

The **cosine** function: $\cos(\theta) = \frac{x}{r}$

In this section, we will explore these functions using both circles and right triangles. In future sections we will take a closer look at the behavior and characteristics of the sine and cosine functions.

Example 1: The point (3, 4) is on the circle of radius 5 at some angle θ. Find $\cos(\theta)$ and $\sin(\theta)$.

$\cos(\theta) = \frac{x}{r}$ Using the formulas evaluate each

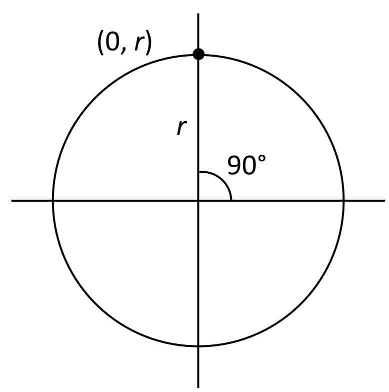
$\sin(\theta) = \frac{y}{r}$

$\cos(\theta) = \frac{3}{5}$ Final answer

$\sin(\theta) = \frac{4}{5}$

There are a few cosine and sine values which we can determine fairly easily because the corresponding point on the circle falls on the x or y axis.

Example 2: Find $\cos(90°)$ and $\sin(90°)$



On any circle, the terminal side of a 90 degree angle points straight up, so the coordinates of the corresponding point on the circle would be (0, r). Using our definitions of cosine and sine,

$$\cos(90) = \frac{x}{r} = \frac{0}{r} = 0$$

$$\sin(90) = \frac{y}{r} = \frac{r}{r} = 1$$

Notice that the definitions above can also be stated as:

Coordinates of the Point on a Circle at a Given Angle

On a circle of radius r at an angle of θ, we can find the **coordinates of the point** $(\boldsymbol{x}, \boldsymbol{y})$ at that angle using

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

On a unit circle, a circle with radius 1, $x = \cos(\theta)$ and $y = \sin(\theta)$.

Utilizing the basic equation for a circle centered at the origin, $x^2 + y^2 = r^2$, combined with the relationships above, we can establish a new identity.

$x^2 + y^2 = r^2$	Substituting the relations above,
$(r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2$	Simplifying,
$r^2(\cos(\theta))^2 + r^2(\sin(\theta))^2 = r^2$	Dividing by r^2
$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$	Or, using shorthand notation
$\cos^2(\theta) + \sin^2(\theta) = 1$	Rewritten in "common" form
$\sin^2(\theta) + \cos^2(\theta) = 1$	Pythagorean Identity!

Here $\cos^2(\theta)$ is a commonly used shorthand notation for $(\cos(\theta))^2$. Be aware that many calculators and computers do not understand the shorthand notation.

We can relate the Pythagorean Theorem $a^2 + b^2 = c^2$ to the basic equation of a circle $x^2 + y^2 = r^2$, which we have now used to arrive at the Pythagorean Identity.

Pythagorean Identity

The **Pythagorean Identity**. For any angle θ, $\sin^2(\theta) + \cos^2(\theta) = 1$.

One use of this identity is that it helps us to find a cosine value of an angle if we know the sine value of that angle or vice versa. However, since the equation will yield two possible values, we will need to utilize additional knowledge of the angle to help us find the desired value.

Example 3: If $\sin(\theta) = \frac{3}{7}$ and θ is in the second quadrant, find $\cos(\theta)$.

$\sin^2(\theta) + \cos^2(\theta) = 1$ Substitute known value

$\left(\frac{3}{7}\right)^2 + \cos^2(\theta) = 1$ Square fraction

$\frac{9}{49} + \cos^2(\theta) = 1$ Subtract $\frac{9}{49}$

$\cos^2(\theta) = \frac{40}{49}$ Square root

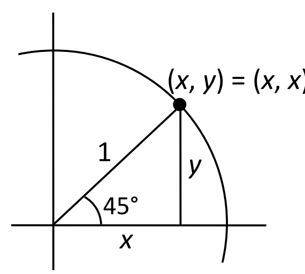
$\cos(\theta) = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}$ Angle is in second quadrant, x is negative

$\cos(\theta) = -\frac{2\sqrt{10}}{7}$ Final answer

Values for Sine and Cosine

At this point, you may have noticed that we haven't found any cosine or sine values from angles not on an axis. To do this, we will need to utilize our knowledge of triangles.

First, consider a point on a circle at an angle of 45 degrees, or $\frac{\pi}{4}$. At this angle, the x and y coordinates of the corresponding point on the circle will be equal because 45 degrees divides the first quadrant in half. Since the x and y values will be the same, the sine and cosine values will also be equal. Utilizing the Pythagorean Identity,



$\sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = 1$ As sin and cos are equal, replace sin with cos

$\cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = 1$ Add like terms

$2\cos^2\left(\frac{\pi}{4}\right) = 1$ Divide

$\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$ Square root (first quadrant, positive x)

$$\cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$ Rationalize denominator

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$ As sin and cos are equal we also know

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

The (x, y) coordinates for a point on a circle of radius 1 at an angle of 45 degrees are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

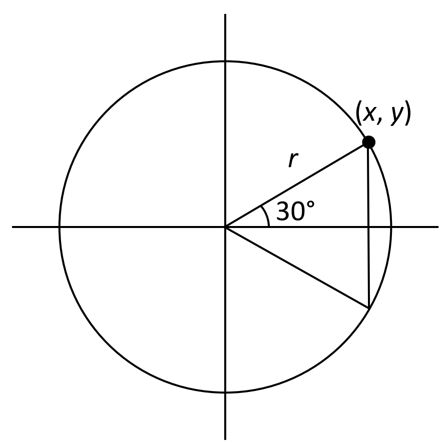
Example 4: Find the coordinates of the point on a circle of radius 6 at an angle of $\frac{\pi}{4}$.

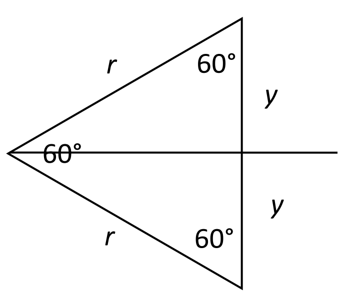
$$r = 6, \theta = \frac{\pi}{4}$$ Using $x = r\cos(\theta)$ and $y = r\sin(\theta)$

Final answer

$$x = 6\cos\left(\frac{\pi}{4}\right) = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$
$$y = 6\sin\left(\frac{\pi}{4}\right) = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$

Next, we will find the cosine and sine at an angle of 30 degrees, or $\frac{\pi}{6}$. To do this, we will first draw a triangle inside a circle with one side at an angle of 30 degrees, and another at an angle of -30 degrees. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60 degrees.





Since all the angles are equal, the sides will all be equal as well. The vertical line has length $2y$, and since the sides are all equal we can conclude that $2y = r$, or $y = \frac{r}{2}$. Using this, we can find the sine value:

$$\sin\left(\frac{\pi}{6}\right) = \frac{y}{r} = \frac{\frac{r}{2}}{r} = \frac{r}{2} \cdot \frac{1}{r} = \frac{1}{2}$$

Using the Pythagorean Identity, we can find the cosine value:

$$\sin^2\left(\frac{\pi}{6}\right) + \cos^2\left(\frac{\pi}{6}\right) = 1$$ Substitute $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\left(\frac{1}{2}\right)^2 + \cos^2\left(\frac{\pi}{6}\right) = 1$$ Square fraction

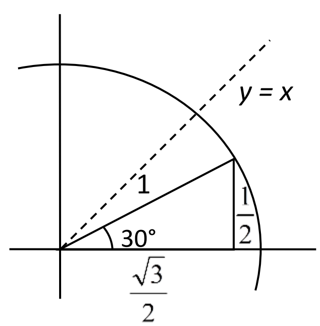
$$\frac{1}{4} + \cos^2\left(\frac{\pi}{6}\right) = 1$$ Subtract $\frac{1}{4}$

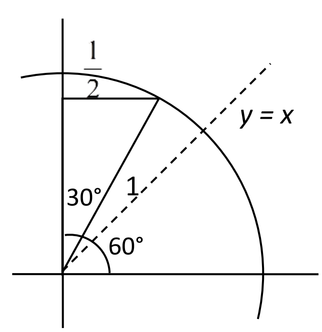
$$\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}$$ Square root (first quadrant, positive x)

$$\cos\left(\frac{\pi}{6}\right) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30 degrees are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

By drawing a triangle inside the unit circle with a 30 degree angle and reflecting it over the line $y = x$, we can find the cosine and sine for 60 degrees, or $\frac{\pi}{3}$, without any additional work.





By this symmetry, we can see the coordinates of the point on the unit circle at an angle of 60 degrees will be $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, giving

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

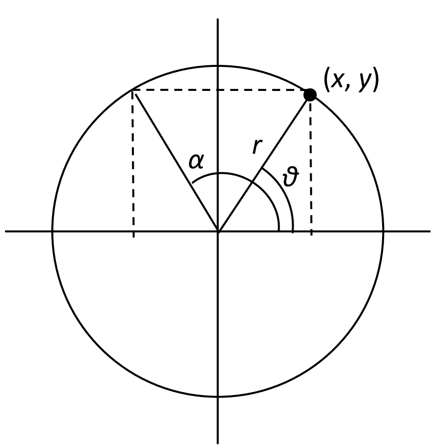
We have now found the cosine and sine values for all of the commonly encountered angles in the first quadrant of the unit circle.

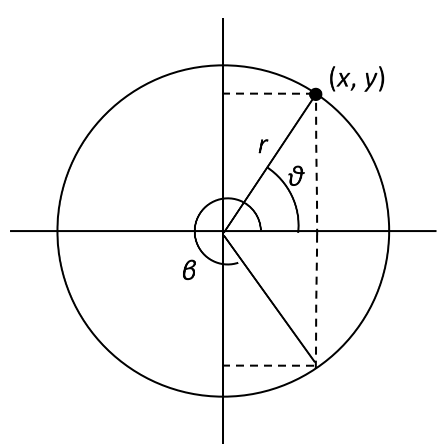
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Degrees	0°	30°	45°	60°	90°
Cos θ	$\sqrt{\frac{4}{4}} = 1$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{0}{4}} = 0$
Sin θ	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}} = 1$

For any given angle in the first quadrant, there will be an angle in another quadrant with the same sine value, and yet another angle in yet another quadrant with the same cosine value. Since the sine value is the y coordinate on the unit circle, the other angle with the same sine will share the same y value, but have the opposite x value. Likewise, the angle with the same cosine will share the same x value, but have the opposite y value.

As shown here, angle α has the same sine value as angle θ; the cosine values would be opposites. The angle β has the same cosine value as the angle; the sine values would be opposites.

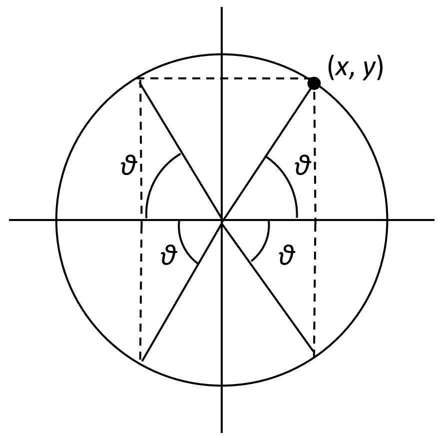
$\sin(\theta) = \sin(\alpha)$ and $\cos(\theta) = -\cos(\alpha)$ $\qquad$ $\sin(\theta) = -\sin(\beta)$ and $\cos(\theta) = \cos(\beta)$





It is important to notice the relationship between the angles. If, from the angle, you measured the smallest angle to the horizontal axis, all would have the same measure in absolute value. We say that all these angles have a **reference angle** of θ

Reference Angle



An angle's **reference angle** is the size of the smallest angle to the horizontal axis.

A reference angle is always an angle between 0 and 90 degrees, or 0 and $\frac{\pi}{2}$ radians.

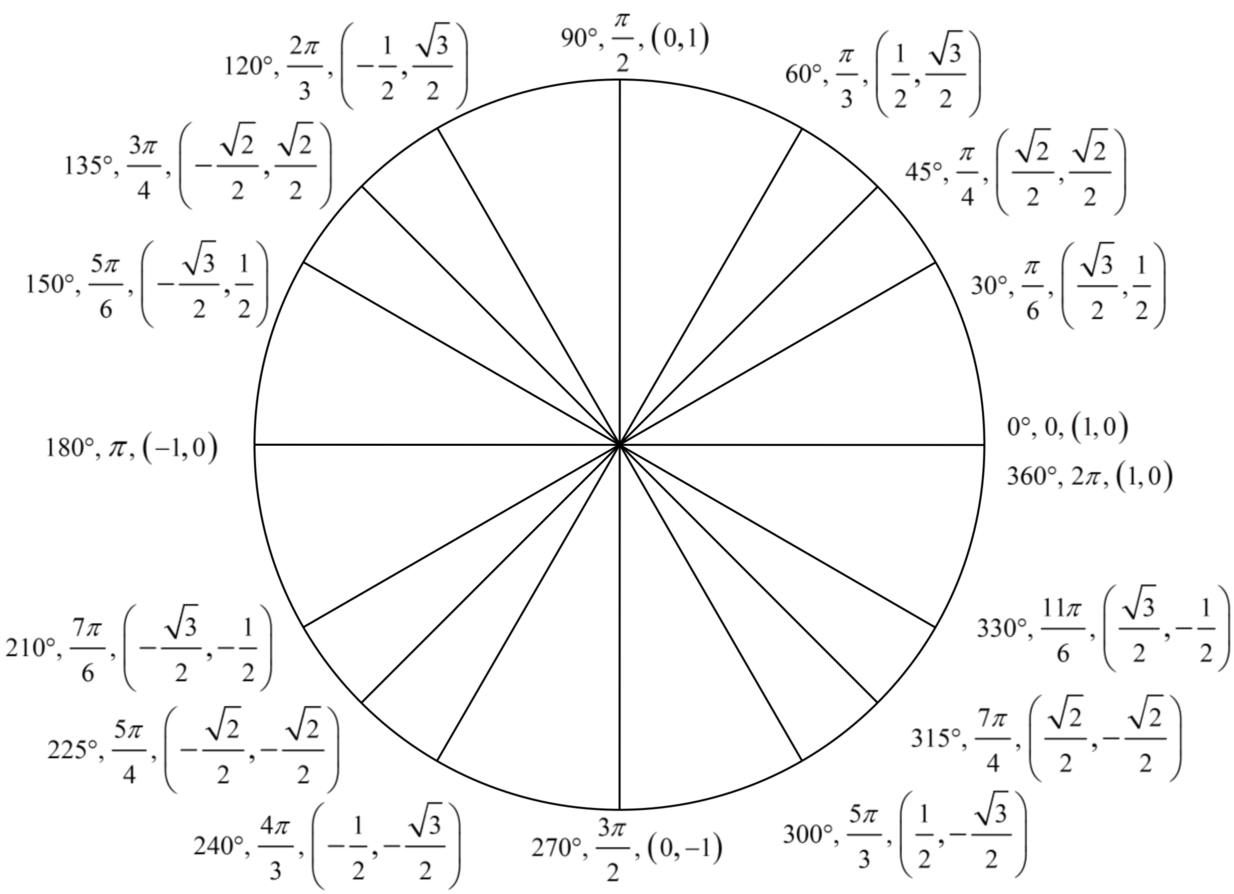
Angles share the same cosine and sine values as their reference angles, except for signs (positive or negative) which can be determined from the quadrant of the angle.

Example 5: Find the reference angle of 150 degrees. Use it to find $\cos(150)$ and $\sin(150)$.

$150°$	30º short of horizontal axis at 180º
Reference angle: 30º	Find sin and cos of this angle
$\sin(30) = \frac{1}{2}, \cos(30) = \frac{\sqrt{3}}{2}$	150 is in second quadrant, x is negative, y is positive
$\sin(150) = \frac{1}{2}, \cos(150) = -\frac{\sqrt{3}}{2}$	Final answer

The (x, y) coordinates for the point on a unit circle at an angle of $150°$ are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Using symmetry and reference angles, we can fill in cosine and sine values at the rest of the special angles on the unit circle. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant!



Example 6: Find the coordinates of the point on a circle of radius 12 at an angle of $\frac{7\pi}{6}$.

Note that this angle is in the third quadrant, where both x and y are negative. Keeping this in mind can help you check your signs of the sine and cosine function.

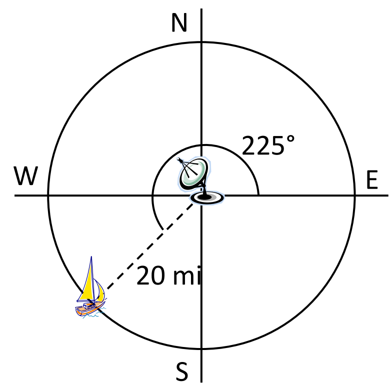
$$x = 12\cos\left(\frac{7\pi}{6}\right) = 12\left(-\frac{\sqrt{3}}{2}\right) = -6\sqrt{3}$$

$$y = 12\sin\left(\frac{7\pi}{6}\right) = 12\left(-\frac{1}{2}\right) = -6$$

The coordinates of the point are $(-6\sqrt{3}, -6)$.

Example 7: We now have the tools to return to the sailboat question posed at the beginning of this section.

A distress signal is sent from a sailboat during a storm, but the transmission is unclear and the rescue boat sitting at the marina cannot determine the sailboat's location. Using high powered radar, they determine the distress signal is coming from a distance of 20 miles at an angle of 225 degrees from the marina. How many miles east/west and north/south of the rescue boat is the stranded sailboat?



$r = 20, \theta = 225$ Use formulas for x and y

$$x = 20\cos(225) = 20\left(-\frac{\sqrt{2}}{2}\right) \approx -14.142$$
$$y = 20\sin(225) = 20\left(-\frac{\sqrt{2}}{2}\right) \approx -14.142$$

Final answer

The sailboat is located 14.142 miles west and 14.142 miles south of the marina.

The special values of sine and cosine in the first quadrant are very useful to know, since knowing them allows you to quickly evaluate the sine and cosine of very common angles without needing to look at a reference or use your calculator. However, scenarios do come up where we need to know the sine and cosine of other angles.

To find the cosine and sine of any other angle, we turn to a computer or calculator. **Be aware**: most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When you evaluate "cos(30)" on your calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode. Most computer software with cosine and sine functions only operates in radian mode.

Example 8: Evaluate the cosine of 20 degrees using a calculator or computer.

On a calculator that can be put in degree mode, you can evaluate this directly to be approximately 0.939693.

On a computer or calculator without degree mode, you would first need to convert the angle to radians, or equivalently evaluate the expression $\cos\left(20 \cdot \frac{\pi}{180}\right)$.

5.4 Points on Circles Practice

1. Find the quadrant in which the terminal point determined by t lies if

   a. $\sin(t) < 0$ and $\cos(t) < 0$ b. $\sin(t) > 0$ and $\cos(t) < 0$

2. Find the quadrant in which the terminal point determined by t lies if

   a. $\sin(t) < 0$ and $\cos(t) > 0$ b. $\sin(t) > 0$ and $\cos(t) > 0$

3. The point P is on the unit circle. If the y-coordinate of P is $\frac{3}{5}$, and P is in quadrant II, find the x coordinate.

4. The point P is on the unit circle. If the x-coordinate of P is $\frac{1}{5}$, and P is in quadrant IV, find the y coordinate.

5. If $\cos(\theta) = \frac{1}{7}$ and θ is in the 4$^{\text{th}}$ quadrant, find $\sin(\theta)$.

6. If $\cos(\theta) = \frac{2}{9}$ and θ is in the 1$^{\text{st}}$ quadrant, find $\sin(\theta)$.

7. If $\sin(\theta) = \frac{3}{8}$ and θ is in the 2$^{\text{nd}}$ quadrant, find $\cos(\theta)$.

8. If $\sin(\theta) = -\frac{1}{4}$ and θ is in the 3$^{\text{rd}}$ quadrant, find $\cos(\theta)$.

9. For each of the following angles, find the reference angle and which quadrant the angle lies in. Then compute sine and cosine of the angle.

   a. 225° b. 300° c. 135° d. 210°

10. For each of the following angles, find the reference angle and which quadrant the angle lies in. Then compute sine and cosine of the angle.

    a. 120° b. 315° c. 250° d. 150°

11. For each of the following angles, find the reference angle and which quadrant the angle lies in. Then compute sine and cosine of the angle.

    a. $\frac{5\pi}{4}$ b. $\frac{7\pi}{6}$ c. $\frac{5\pi}{3}$ d. $\frac{3\pi}{4}$

12. For each of the following angles, find the reference angle and which quadrant the angle lies in. Then compute sine and cosine of the angle.

a. $\frac{4\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{5\pi}{6}$ d. $\frac{7\pi}{4}$

13. Give exact values for $\sin(\theta)$ and $\cos(\theta)$ for each of these angles.

a. $-\frac{3\pi}{4}$ b. $\frac{23\pi}{6}$ c. $-\frac{\pi}{2}$ d. 5π

14. Give exact values for $\sin(\theta)$ and $\cos(\theta)$ for each of these angles.

a. $-\frac{2\pi}{3}$ b. $\frac{17\pi}{4}$ c. $-\frac{\pi}{6}$ d. 10π

15. Find an angle θ with $0° < \theta < 360°$ or $0 < \theta < 2\pi$ that has the same sine value as:

a. $\frac{\pi}{3}$ b. 80° c. 140° d. $\frac{4\pi}{3}$ e. 305°

16. Find an angle θ with $0° < \theta < 360°$ or $0 < \theta < 2\pi$ that has the same sine value as:

a. $\frac{\pi}{4}$ b. 15° c. 160° d. $\frac{7\pi}{6}$ e. 340°

17. Find an angle θ with $0° < \theta < 360°$ or $0 < \theta < 2\pi$ that has the same cosine value as:

a. $\frac{\pi}{3}$ b. 80° c. 140° d. $\frac{4\pi}{3}$ e. 305°

18. Find an angle θ with $0° < \theta < 360°$ or $0 < \theta < 2\pi$ that has the same cosine value as:

a. $\frac{\pi}{4}$ b. 15° c. 160° d. $\frac{7\pi}{6}$ e. 340°

19. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of 220°.

20. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 280°.

21. Marla is running clockwise around a circular track. She runs at a constant speed of 3 meters per second. She takes 46 seconds to complete one lap of the track. From her starting point, it takes her 12 seconds to reach the northernmost point of the track. Impose a coordinate system with the center of the track at the origin, and the northernmost point on the positive y-axis.

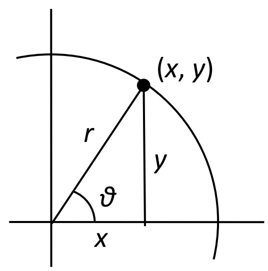
a) Give Marla's coordinates at her starting point.
b) Give Marla's coordinates when she has been running for 10 seconds.
c) Give Marla's coordinates when she has been running for 901.3 seconds.

5.5 Other Trigonometric Functions

In the previous section, we defined the sine and cosine functions as ratios of the sides of a right triangle in a circle. Since the triangle has 3 sides there are 6 possible combinations of ratios. While the sine and cosine are the two prominent ratios that can be formed, there are four others, and together they define the 6 trigonometric functions.

Tangent, Secant, Cosecant, and Cotangent Functions

For the point (x, y) on a circle of radius r at an angle of θ, we can define four additional important functions as the ratios of the sides of the corresponding triangle:



The **tangent** function: $\tan(\theta) = \frac{y}{x}$

The **secant** function: $\sec(\theta) = \frac{r}{x}$

The **cosecant** function: $\csc(\theta) = \frac{r}{y}$

The **cotangent** function: $\cot(\theta) = \frac{x}{y}$

Geometrically, notice that the definition of tangent corresponds with the slope of the line segment between the origin (0, 0) and the point (x, y). This relationship can be very helpful in thinking about tangent values.

You may also notice that the ratios defining the secant, cosecant, and cotangent are the reciprocals of the ratios defining the cosine, sine, and tangent functions, respectively. Additionally, notice that using our results from the last section,

$$\tan(\theta) = \frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}$$

Applying this concept to the other trig functions we can state the other reciprocal identities.

Identities

The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)}, \quad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

These relationships are called **identities**. Identities are statements that are true for all values of the input on which they are defined. Identities are usually something that can be derived from definitions and relationships we already know, similar to how the identities above were derived from the circle relationships of the six trig functions. The Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine. We will discuss the role of identities more after an example.

Example 1: Evaluate $\tan(45°)$

$\tan(45°)$	Rewrite with sine and cosine
$\tan(45°) = \frac{\sin(45°)}{\cos(45°)}$	Recall values for sine and cosine and simplify
$\tan(45°) = \frac{\sin(45°)}{\cos(45°)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$	Final answer

Notice this result is consistent with our interpretation of the tangent value as the slope of the line passing through the origin at the given angle: a line at 45 degrees would indeed have a slope of 1.

Example 2: Evaluate $\sec\left(\frac{5\pi}{6}\right)$

$\sec\left(\frac{5\pi}{6}\right)$	Rewrite with cosine
$\sec\left(\frac{5\pi}{6}\right) = \frac{1}{\cos\left(\frac{5\pi}{6}\right)}$	Evaluate cosine and simplify
$\sec\left(\frac{5\pi}{6}\right) = \frac{1}{\cos\left(\frac{5\pi}{6}\right)} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$	Rationalize denominator
$-\frac{2\sqrt{3}}{3}$	Final answer

Just as we often need to simplify algebraic expressions, it is often also necessary or helpful to simplify trigonometric expressions. To do so, we utilize the definitions and identities we have established.

Example 3: Simplify $\frac{\sec(\theta)}{\tan(\theta)}$.

$\frac{\sec(\theta)}{\tan(\theta)}$	Rewrite in terms of sine and cosine
$\frac{\frac{1}{\cos(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)}}$	Multiply by the reciprocal
$\frac{1}{\cos(\theta)} \cdot \frac{\cos(\theta)}{\sin(\theta)}$	Reduce the cosines
$\frac{1}{\sin(\theta)}$	Simplify using the identity
$\csc(\theta)$	Final answer

By showing that $\frac{\sec(\theta)}{\tan(\theta)}$ can be simplified to $\csc(\theta)$, we have, in fact, established a new identity: that $\frac{\sec(\theta)}{\tan(\theta)} = \csc(\theta)$.

Occasionally a question may ask you to "prove the identity" or "establish the identity." This is the same idea as when an algebra book asks a question like "show that $(x-1)^2 = x^2 - 2x + 1$." In this type of question we must show the algebraic manipulations that demonstrate that the left and right side of the equation are in fact equal. You can think of a "prove the identity" problem as a simplification problem where you *know the answer*: you know what the end goal of the simplification should be, and just need to show the steps to get there.

To prove an identity, in most cases you will start with the expression on one side of the identity and manipulate it using algebra and trigonometric identities until you have simplified it to the expression on the other side of the equation. **Do not** treat the identity like an equation to solve – it isn't! The proof is establishing *if* the two expressions are equal, so we must take care to work with one side at a time rather than applying an operation simultaneously to both sides of the equation.

Example 4: Prove the identity $\frac{1+\cot(\alpha)}{\csc(\alpha)} = \sin(\alpha) + \cos(\alpha)$.

Since the left side seems a bit more complicated, we will start there and simplify the expression until we obtain the right side. We can use the right side as a guide for what might be good steps to make. In this case, the left side involves a fraction while the right side doesn't, which suggests we should look to see if the fraction can be reduced.

Additionally, since the right side involves sine and cosine and the left does not, it suggests that rewriting the cotangent and cosecant using sine and cosine might be a good idea.

$$\frac{1+\cot(\alpha)}{\csc(\alpha)}$$ Rewrite with sine and cosine

$$\frac{1+\dfrac{\cos(\alpha)}{\sin(\alpha)}}{\dfrac{1}{\sin(\alpha)}}$$ Multiply by the reciprocal

$$\left(1+\frac{\cos(\alpha)}{\sin(\alpha)}\right)\sin(\alpha)$$ Distribute

$$\sin(\alpha) + \frac{\cos(\alpha)\sin(\alpha)}{\sin(\alpha)}$$ Reduce

$$\sin(\alpha) + \cos(\alpha)$$ Final answer

Notice that in the second step, we could have combined the 1 and $\frac{\cos(\alpha)}{\sin(\alpha)}$ before inverting and multiplying. It is very common when proving or simplifying identities for there to be more than one way to obtain the same result.

We can also utilize identities we have previously learned, like the Pythagorean Identity, while simplifying or proving identities.

Example 5: Establish the identity $\frac{\cos^2(\theta)}{1+\sin(\theta)} = 1 - \sin(\theta)$.

Since the left side of the identity is more complicated, it makes sense to start there. To simplify this, we will have to reduce the fraction, which would require the numerator to have a factor in common with the denominator. Additionally, we notice that the right side only involves sine. Both of these suggest that we need to convert the cosine into something involving sine.

Recall the Pythagorean Identity told us $\sin^2(\theta) + \cos^2(\theta) = 1$. By moving one of the trig functions to the other side, we can establish:

$$\sin^2(\theta) = 1 - \cos^2(\theta) \quad \text{and} \quad \cos^2(\theta) = 1 - \sin^2(\theta)$$

Utilizing this, we now can establish the identity. We start on one side and manipulate:

$$\frac{\cos^2(\theta)}{1+\sin(\theta)}$$ Replace $\cos^2(\theta)$ with Pythagorean identity

$$\frac{1-\sin^2(\theta)}{1+\sin(\theta)}$$ Factor the numerator

$$\frac{(1+\sin(\theta))(1-\sin(\theta))}{1+\sin(\theta)}$$ Reducing the like factors

$$1 - \sin(\theta)$$ Final answer

We can also build new identities from previously established identities. For example, if we divide both sides of the Pythagorean Identity by cosine squared (which is allowed since we've already shown the identity is true), we find a new identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Divide both sides by $\cos^2(\theta)$

$$\frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

Divide each term on the right

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

Simplify using definition of $\tan(\theta)$ and $\sec(\theta)$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

New identity

Similarly, by dividing the original Pythagorean identity by $\sin^2(\theta)$ it can be shown that $1 + \cot^2(\theta) = \csc^2(\theta)$

Alternate forms of the Pythagorean Identity

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Example 6: If $\tan(\theta) = \frac{2}{7}$ and θ is in the 3rd quadrant, find $\cos(\theta)$.

There are two approaches to this problem, both of which work equally well.

<u>Approach 1</u>

Since $\tan(\theta) = \frac{y}{x}$ and the angle is in the third quadrant, we can imagine a triangle in a circle of some radius so that the point on the circle is $(-7, -2)$. Using the Pythagorean Theorem, we can find the radius of the circle:

$$(-7)^2 + (-2)^2 = r^2$$

Simplify and solve for r

$$r = \sqrt{53}$$

Now find $\cos(\theta) = \frac{x}{r}$

$$\cos(\theta) = \frac{x}{r} = -\frac{7}{\sqrt{57}} = -\frac{7\sqrt{53}}{53}$$

Final answer

Approach 2

Using the $\tan^2(\theta) + 1 = \sec^2(\theta)$ form of the Pythagorean Identity with the known tangent,

$\tan^2(\theta) + 1 = \sec^2(\theta)$	Substitute $\frac{2}{7}$ for $\tan(\theta)$
$\left(\frac{2}{7}\right)^2 + 1 = \sec^2(\theta)$	Simplify left side
$\frac{53}{49} = \sec^2(\theta)$	Square root
$\pm\frac{\sqrt{53}}{7} = \sec(\theta)$	Cosine is the reciprocal of secant
$\pm\frac{7}{\sqrt{53}} = \cos(\theta)$	Third quadrant, x or cosine is negative
$\cos(\theta) = -\frac{7}{\sqrt{53}}$	Rationalize denominator
$\cos(\theta) = -\frac{7\sqrt{53}}{53}$	Final answer

5.5 Other Trigonometric Functions Practice

1. If $\theta = \frac{\pi}{4}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

2. If $\theta = \frac{7\pi}{4}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

3. If $\theta = \frac{5\pi}{6}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

4. If $\theta = \frac{\pi}{6}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

5. If $\theta = \frac{2\pi}{3}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

6. If $\theta = \frac{4\pi}{3}$, find exact values for $\sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

7. Evaluate: a. $\sec(135°)$ b. $\csc(210°)$ c. $\tan(60°)$ d. $\cot(225°)$

8. Evaluate: a. $\sec(30°)$ b. $\csc(315°)$ c. $\tan(135°)$ d. $\cot(150°)$

9. If $\sin(\theta) = \frac{3}{4}$, and θ is in quadrant II, find $\cos(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

10. If $\sin(\theta) = \frac{2}{7}$, and θ is in quadrant II, find $\cos(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

11. If $\cos(\theta) = -\frac{1}{3}$, and θ is in quadrant III, find $\sin(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

12. If $\cos(\theta) = \frac{1}{5}$, and θ is in quadrant I, find $\sin(\theta), \sec(\theta), \csc(\theta), \tan(\theta), \cot(\theta)$.

13. If $\tan(\theta) = \frac{12}{5}$, and $0 \le \theta < \frac{\pi}{2}$, find $\sin(\theta), \cos(\theta), \sec(\theta), \csc(\theta), \cot(\theta)$.

14. If $\tan(\theta) = 4$, and $0 \le \theta < \frac{\pi}{2}$, find $\sin(\theta), \cos(\theta), \sec(\theta), \csc(\theta), \cot(\theta)$.

15. Use a calculator to find sine, cosine, and tangent of the following values:

    a. 0.15 b. 4 c. 70° d. 283°

16. Use a calculator to find sine, cosine, and tangent of the following values:

    a. 0.5 b. 5.2 c. 10° d. 195°

Simplify each of the following to an expression involving a single trig function with no fractions.

17. $\csc(t)\tan(t)$

18. $\cos(t)\csc(t)$

19. $\dfrac{\sec(t)}{\csc(t)}$

20. $\dfrac{\cot(t)}{\csc(t)}$

21. $\dfrac{\sec(t)-\cos(t)}{\sin(t)}$

22. $\dfrac{\tan(t)}{\sec(t)-\cos(t)}$

23. $\dfrac{1+\cot(t)}{1+\tan(t)}$

24. $\dfrac{1+\sin(t)}{1+\csc(t)}$

25. $\dfrac{\sin^2(t)+\cos^2(t)}{\cos^2(t)}$

26. $\dfrac{1-\sin^2(t)}{\sin^2(t)}$

Prove the identities.

27. $\dfrac{\sin^2(\theta)}{1+\cos(\theta)}=1-\cos(\theta)$

28. $\tan^2(t)=\dfrac{1}{\cos^2(t)}-1$

29. $\sec(a)-\cos(a)=\sin(a)\tan(a)$

30. $\dfrac{1+\tan^2(b)}{\tan^2(b)}=\csc^2(b)$

31. $\dfrac{\csc^2(x)-\sin^2(x)}{\csc(x)+\sin(x)}=\cos(x)\cot(x)$

32. $\dfrac{\sin(\theta)-\cos(\theta)}{\sec(\theta)-\csc(\theta)}=\sin(\theta)\cos(\theta)$

33. $\dfrac{\csc^2(\alpha)-1}{\csc^2(\alpha)-\csc(\alpha)}=1+\sin(\alpha)$

34. $1+\cot(x)=\cos(x)\,(\sec(x)+\csc(x))$

35. $\dfrac{1+\cos(u)}{\sin(u)}=\dfrac{\sin(u)}{1-\cos(u)}$

36. $2\sec^2(t)=\dfrac{1-\sin(t)}{\cos^2(t)}+\dfrac{1}{1-\sin(t)}$

37. $\dfrac{\sin^2(\gamma)-\cos^2(\gamma)}{\sin(\gamma)-\cos(\gamma)}=\sin(\gamma)+\cos(\gamma)$

38. $\dfrac{(1+\cos(A))(1-\cos(A))}{\sin(A)}=\sin(A)$

39. $(\sec(\theta)+\tan(\theta))(1-\sin(\theta))$
$=\cos(\theta)$

40. $\dfrac{\tan(\theta)+\cos(\theta)}{\sin(\theta)}=\sec(\theta)+\cot(\theta)$

5.6 Graphs of Trig Functions

The London Eye is a huge Ferris wheel with diameter 135 meters (443 feet) in London, England, which completes one rotation every 30 minutes. When we look at the behavior of this Ferris wheel it is clear that it completes 1 cycle, or 1 revolution, and then repeats this revolution over and over again.

This is an example of a periodic function, because the Ferris wheel repeats its revolution or one cycle every 30 minutes, and so we say it has a period of 30 minutes.

In this section, we will work to sketch a graph of a rider's height above the ground over time and express this height as a function of time.

Periodic Functions

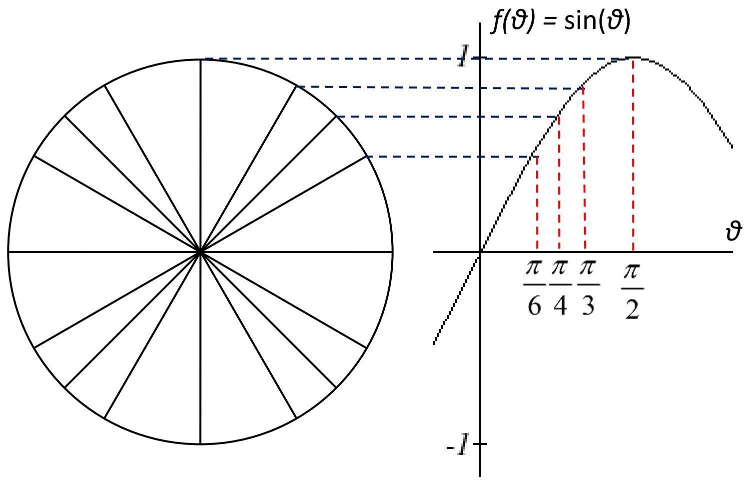
A **periodic function** is a function for which a specific horizontal shift, P, results in the original function: $f(x+P) = f(x)$ for all values of x. When this occurs we call the smallest such horizontal shift with $P > 0$ the **period** of the function.

You might immediately guess that there is a connection here to finding points on a circle, since the height above ground would correspond to the y value of a point on the circle. We can determine the y value by using the sine function. To get a better sense of this function's behavior, we can create a table of values we know, and use them to sketch a graph of the sine and cosine functions.

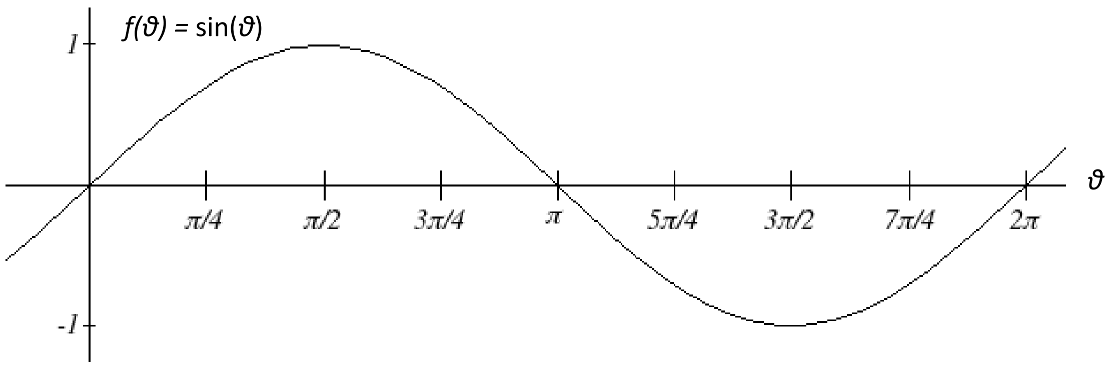
Listing some of the values for sine and cosine on a unit circle,

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

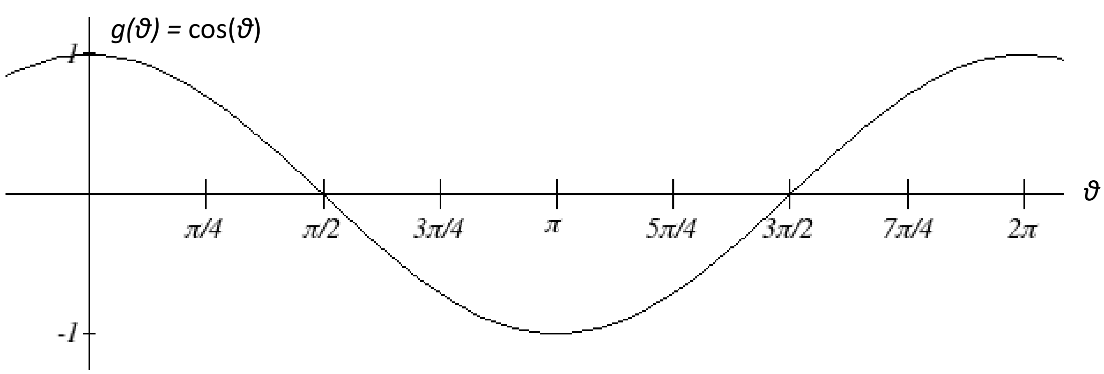
Here you can see how for each angle, we use the y value of the point on the circle to determine the output value of the sine function.



Plotting more points gives the full shape of the sine and cosine functions.



Notice how the sine values are positive between 0 and π, which correspond to the values of sine in quadrants 1 and 2 on the unit circle, and the sine values are negative between π and 2π, corresponding to quadrants 3 and 4.



Like the sine function we can track the value of the cosine function through the 4 quadrants of the unit circle as we place it on a graph.

Both of these functions are defined for all real numbers, since we can evaluate the sine and cosine of any angle. By thinking of sine and cosine as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1,1]$.

Domain and Range of Sine and Cosine

The domain of sine and cosine is all real numbers, $(-\infty, \infty)$.

The range of sine and cosine is the interval $[-1,1]$.

Both these graphs are called **sinusoidal** graphs.

In both graphs, the shape of the graph begins repeating after 2π. Indeed, since any coterminal angles will have the same sine and cosine values, we could conclude that $\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$.

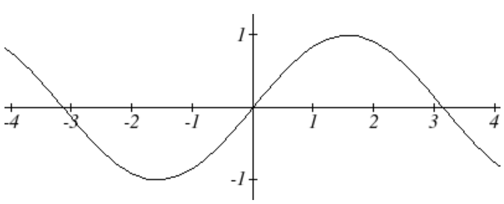
In other words, if you were to shift either graph horizontally by 2π, the resulting shape would be identical to the original function. Sinusoidal functions are a specific type of periodic function.

Period of Sine and Cosine

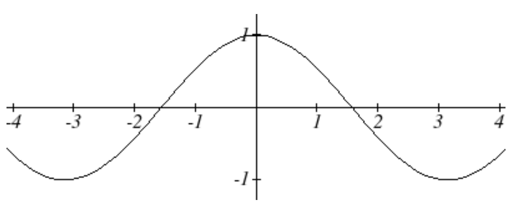
The periods of the sine and cosine functions are both 2π.

Looking at these functions on a domain centered at the vertical axis helps reveal symmetries.

sine



cosine



The sine function is symmetric about the origin, the same symmetry the cubic function has, making it an odd function. The cosine function is clearly symmetric about the y axis, the same symmetry as the quadratic function, making it an even function.

Negative Angle Identities

The sine is an odd function, symmetric about the *origin*, so $\sin(-\theta) = -\sin(\theta)$.

The cosine is an even function, symmetric about the y-axis, so $\cos(-\theta) = \cos(\theta)$.

These identities can be used, among other purposes, for helping with simplification and proving identities.

You may recall the cofunction identity from an earlier section: $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$. Graphically, this tells us that the sine and cosine graphs are horizontal transformations of each other. We can prove this by using the cofunction identity and the negative angle identity for cosine.

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(-\theta + \frac{\pi}{2}\right) = \cos\left(-\left(\theta - \frac{\pi}{2}\right)\right) = \cos\left(\theta - \frac{\pi}{2}\right)$$

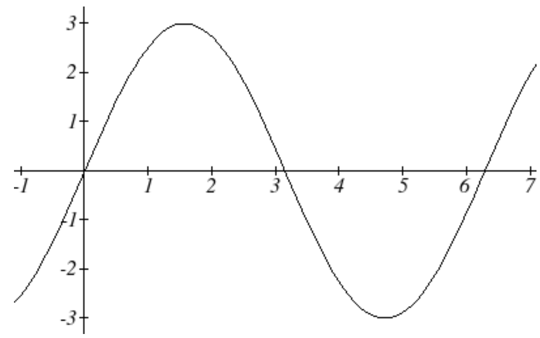
Now we can clearly see that if we horizontally shift the cosine function to the right by $\frac{\pi}{2}$ we get the sine function.

Remember this shift is not representing the period of the function. It only shows that the cosine and sine function are transformations of each other.

Example 1: Simplify $\frac{\sin(-\theta)}{\tan(\theta)}$.

$\frac{\sin(-\theta)}{\tan(\theta)}$	Use the even/odd identity
$\frac{-\sin(\theta)}{\tan(\theta)}$	Rewrite the tangent
$\frac{-\sin(\theta)}{\frac{\sin(\theta)}{\cos(\theta)}}$	Multiply by the reciprocal
$-\sin(\theta)\cdot\frac{\cos(\theta)}{\sin(\theta)}$	Reduce
$-\cos(\theta)$	Final answer

Transforming Sine and Cosine



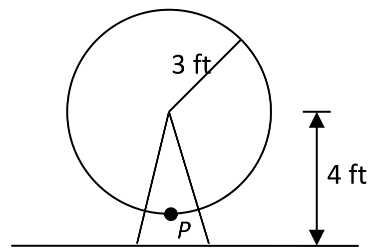
Example 2: A point rotates around a circle of radius 3. Sketch a graph of the *y* coordinate of the point.

Recall that for a point on a circle of radius r, the y coordinate of the point is $y = rsin(\theta)$, so in this case, we get the equation $y(\theta) = 3\sin(\theta)$.

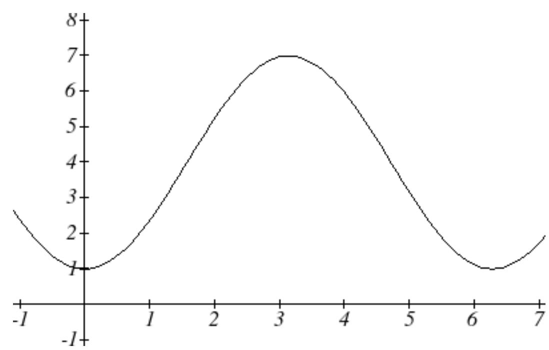
The constant 3 causes a vertical stretch of the *y* values of the function by a factor of 3.

Notice that the period of the function does not change.

Since the outputs of the graph will now oscillate between -3 and 3, we say that the **amplitude** of the sine wave is 3.



Example 3: A circle with radius 3 feet is mounted with its center 4 feet off the ground. The point closest to the ground is labeled *P*. Sketch a graph of the height above ground of the point *P* as the circle is rotated, then find a function that gives the height in terms of the angle of rotation.



Sketching the height, we note that it will start 1 foot above the ground, then increase up to 7 feet above the ground, and continue to oscillate 3 feet above and below the center value of 4 feet.

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other.

We decide to use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of one, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together,

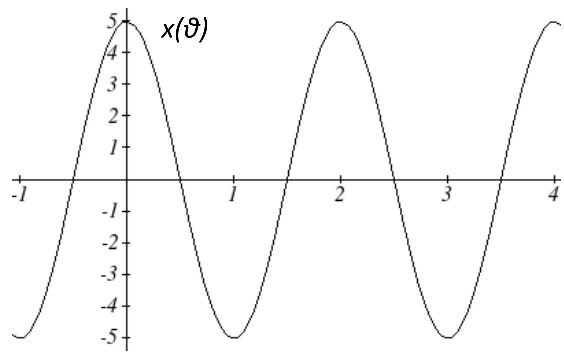
$$h(\theta) = -3\cos(\theta) + 4$$

To answer the Ferris wheel problem at the beginning of the section, we need to be able to express our sine and cosine functions at inputs of time. To do so, we will utilize composition. Since the sine function takes an input of an angle, we will look for a function that takes time as an input and outputs an angle. If we can find a suitable $\theta(t)$ function, then we can compose this with our $f(\theta) = \cos(\theta)$ function to obtain a sinusoidal function of time: $f(t) = \cos(\theta(t))$.

Example 4: A point completes 1 revolution every 2 minutes around a circle of radius 5. Find the x coordinate of the point as a function of time, if it starts at (5, 0).

Normally, we would express the x coordinate of a point on a unit circle using $x = r\cos(\theta)$, here we write the function $x(\theta) = 5\cos(\theta)$.

The rotation rate of 1 revolution every 2 minutes is an angular velocity. We can use this rate to find a formula for the angle as a function of time. The point begins at an angle of 0. Since the point rotates 1 revolution is 2π radians every 2 minutes, it rotates π radians every minute. After t minutes, it will have rotated:



$$\theta(t) = \pi t \text{ radians}$$

Composing this with the cosine function, we obtain a function of time.

$$x(t) = 5\cos\big(\theta(t)\big) = 5\cos(\pi t)$$

Notice that this composition has the effect of a horizontal compression, changing the period of the function.

To see how the period relates to the stretch or compression coefficient B in the equation $f(t) = \sin(Bt)$, note that the period will be the time it takes to complete one full revolution of a circle. If a point takes P minutes to complete 1 revolution, then the angular velocity is $\frac{2\pi \text{ radians}}{P \text{ mintues}}$. Then $\theta(t) = \frac{2\pi}{P}t$. Composing with a sine function, $f(t) = \sin\big(\theta(t)\big) = \sin(\frac{2\pi}{P}t)$

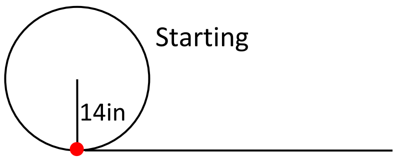
From this, we can determine the relationship between the coefficient B and the period: $B = \frac{2\pi}{P}$. Notice that the stretch or compression coefficient B is a ratio of the "normal period of a sinusoidal function" to the "new period." If we know the stretch or compression coefficient B, we can solve for the "new period": $P = \frac{2\pi}{B}$.

Example 5: What is the period of the function $f(t) = \sin\left(\frac{\pi}{6}t\right)$?

$\sin\left(\frac{\pi}{6}t\right)$	Use formula $P = \frac{2\pi}{B}$ where $B = \frac{\pi}{6}$
$\frac{2\pi}{\frac{\pi}{6}}$	Multiply by reciprocal
$2\pi \cdot \frac{6}{\pi}$	Simplify
12	Final answer

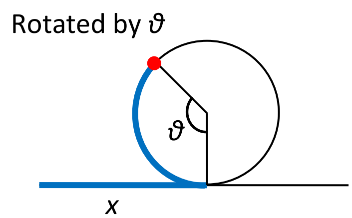
While it is common to compose sine or cosine with functions involving time, the composition can be done so that the input represents any reasonable quantity.

Example 6: A bicycle wheel with radius 14 inches has the bottom-most point on the wheel marked in red. The wheel then begins rolling down the street. Write a formula for the height above ground of the red point after the bicycle has travelled x inches.



The height of the point begins at the lowest value, 0, increases to the highest value of 28 inches, and continues to oscillate above and below a center height of 14 inches. In terms of the angle of rotation, θ:

$$h(\theta) = -14\cos(\theta) + 14$$



In this case, x is representing a linear distance the wheel has travelled, corresponding to an arclength along the circle. Since arc length and angle can be related by $s = r\theta$, in this case we can write $x = 14\theta$, which allows us to express the angle in terms of x: $\theta(x) = \frac{x}{14}$

Composing this with our cosine-based function from above,

$$h(x) = h(\theta(x)) = -14\cos\left(\frac{x}{14}\right) + 14 = -14\cos\left(\frac{1}{14}x\right) + 14$$

The period of this function would be $= \frac{2\pi}{B} = \frac{2\pi}{\frac{1}{14}} = 2\pi \cdot 14 = 28\pi$, the circumference of the circle. This makes sense – the wheel completes one full revolution after the bicycle has travelled a distance equivalent to the circumference of the wheel.

Summarizing our transformations so far:

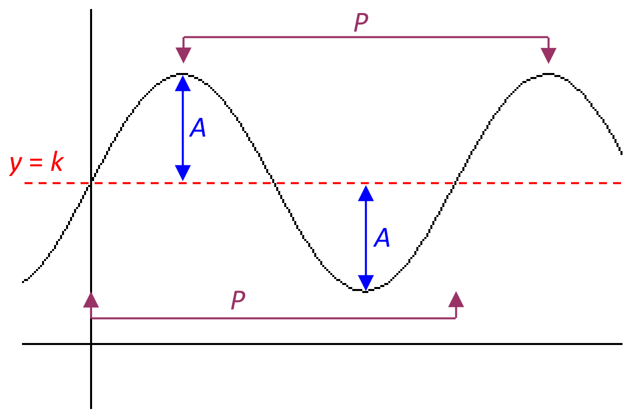
Transformations of Sine and Cosine

Given an equation in the form $f(t) = A\sin(Bt) + k$ or $f(t) = A\cos(Bt) + k$

A is the vertical stretch, and is the **amplitude** of the function.

B is the horizontal stretch/compression, and is related to the **period, *P***, by $P = \frac{2\pi}{B}$.

k is the vertical shift



Example 7: Determine the vertical shift, amplitude, and period of the function
$f(t) = 3\sin(2t) + 1$.

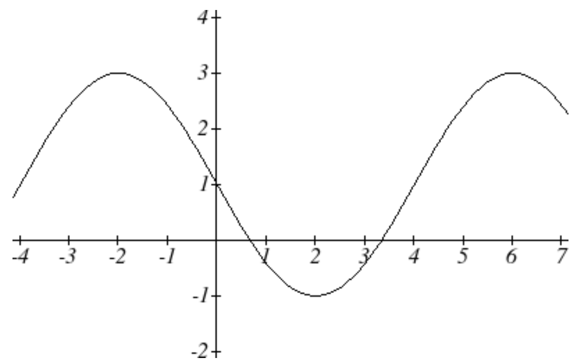
The amplitude is 3

The period is $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

The vertical shift is up 1.

Amplitude, vertical shift, and period, when combined with vertical reflections across the horizontal axis, allow us to write equations for a variety of sinusoidal situations.

Example 8: Find a formula for the sinusoidal function graphed here.



The graph oscillates from a low of -1 to a high of 3, putting the vertical shift at up 1, halfway between.

The amplitude will be 2, the distance from the vertical shift to the highest value (or lowest value) of the graph.

The period of the graph is 8. We can measure this from the first peak at $x = -2$ to the second at $x = 6$. Since the period is 8, the stretch/compression factor we will use will be

$$B = \frac{2\pi}{P} = \frac{2\pi}{8} = \frac{\pi}{4}$$

At $x = 0$, the graph is at the middle value, which tells us the graph can most easily be represented as a sine function. Since the graph then decreases, this must be a vertical reflection of the sine function. Putting this all together,

$$f(t) = -2\sin\left(\frac{\pi}{4}t\right) + 1$$

With these transformations, we are ready to answer the Ferris wheel problem from the beginning of the section.

Example 9: The London Eye is a huge Ferris wheel with diameter 135 meters (443 feet) in London, England, which completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

With a diameter of 135 meters, the wheel has a radius of 67.5 meters. The height will oscillate with amplitude of 67.5 meters above and below the center.

Passengers board 2 meters above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ meters above ground level. The vertical shift of the oscillation will be at 69.5 meters.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with period of 30 minutes.

Lastly, since the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a reflected cosine curve with respect to the horizontal axis.

Putting these together:

Amplitude: 67.5

Midline: 69.5

Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$

Shape: $-\cos$ function

An equation for the rider's height would be

$$h(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

While these transformations are sufficient to represent many situations, occasionally we encounter a sinusoidal function that does not have a vertical intercept at the lowest point, highest point, or midline. In these cases, we need to use horizontal shifts. Recall that when the inside of the function is factored, it reveals the horizontal shift.

Horizontal Shifts of Sine and Cosine

Given an equation in the form $f(t) = A\sin\big(B(t-h)\big) + k$ or $f(t) = A\cos\big(B(t-h)\big) + k$

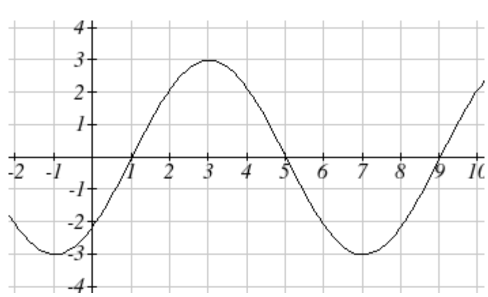
h is the horizontal shift of the function

Example 10: Sketch a graph of $(t) = 3\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right)$.

To reveal the horizontal shift, we first need to factor inside the function:

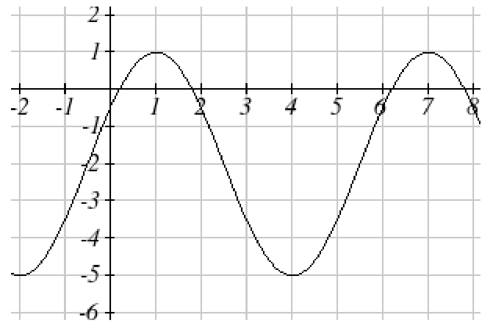
$$f(t) = 3\sin\left(\frac{\pi}{4}(t-1)\right)$$

This graph will have the shape of a sine function, starting at the middle and increasing, with an amplitude of 3. The period of the graph will be $P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$. Finally, the graph will be shifted to the right by 1.



In some physics and mathematics books, you will hear the horizontal shift referred to as **phase shift**. In other physics and mathematics books, they would say the phase shift of the equation above is $\frac{\pi}{4}$, the value in the unfactored form. Because of this ambiguity, we will not use the term phase shift any further, and will only talk about the horizontal shift.

Example 11: Find a formula for the function graphed here.



With highest value at 1 and lowest value at -5, the middle will be halfway between at -2.

The distance from the middle to the highest or lowest value gives an amplitude of 3.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. This gives $B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}$.

For the shape and shift, we have more than one option. We could either write this as:

A cosine function shifted 1 to the right

A negative cosine function shifted 2 to the left

A sine function shifted $\frac{1}{2}$ to the left

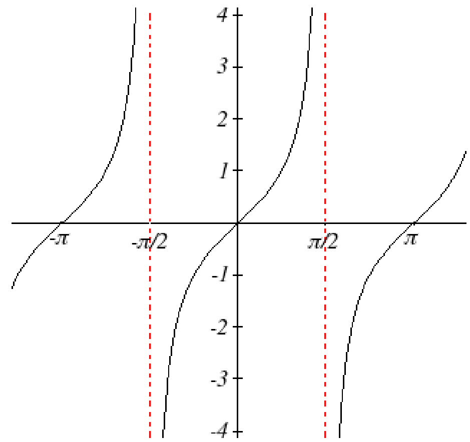
A negative sine function shifted 2.5 to the right

While any of these would be fine, the cosine shifts are easier to work with than the sine shifts in this case, because they involve integer values. Writing these:

$$y(x) = 3\cos\left(\frac{\pi}{3}(x-1)\right) - 2 \quad \text{or} \quad y(x) = -3\cos\left(\frac{\pi}{3}(x+2)\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

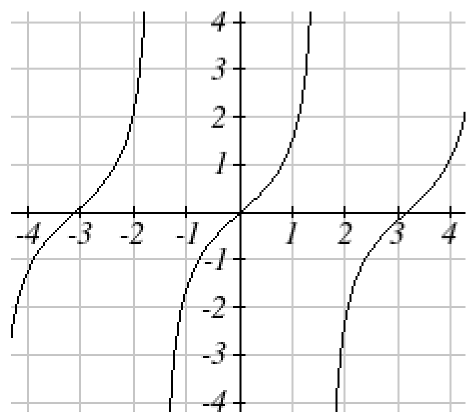
Next, we will explore the graphs of the other four trigonometric functions. We'll begin with the tangent function. Recall that we defined tangent as $\frac{y}{x}$ or $\frac{\sin}{\cos}$, so you can think of the tangent as the slope of a line through the origin making the given angle with the positive x axis. At an angle of 0, the line would be horizontal with a slope of zero. As the angle increases towards $\frac{\pi}{2}$, the slope increases more and more. At an angle of $\frac{\pi}{2}$, the line would be vertical and the slope would be undefined. Immediately past $\frac{\pi}{2}$, the line would have a steep negative slope, giving a large negative tangent value. There is a break in the function at $\frac{\pi}{2}$, where the tangent value jumps from large positive to large negative.



We can use these ideas along with the definition of tangent to sketch a graph. Since tangent is defined as $\frac{\sin}{\cos}$, we can determine that tangent will be zero when sine is zero: at $-\pi, 0, \pi,$ and so on. Likewise, tangent will be undefined when cosine is zero: at $-\frac{\pi}{2}, \frac{\pi}{2}$, and so on.

The tangent is positive from 0 to $\frac{\pi}{2}$ and π to $\frac{3\pi}{2}$, corresponding to quadrants 1 and 3 of the unit circle.

Using technology, we can obtain a graph of tangent on a standard grid.



Notice that the graph appears to repeat itself. For any angle on the circle, there is a second angle with the same slope and tangent value halfway around the circle, so the graph repeats itself with a period of π; we can see one continuous cycle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, before it jumps and repeats itself.

The graph has vertical asymptotes and the tangent is undefined wherever a line at that angle would be vertical: at $\frac{\pi}{2}, \frac{3\pi}{2}$, and so on. While the domain of the function is limited in this way, the range of the function is all real numbers.

Features of the Graph of Tangent

The graph of the tangent function $m(\theta) = \tan(\theta)$

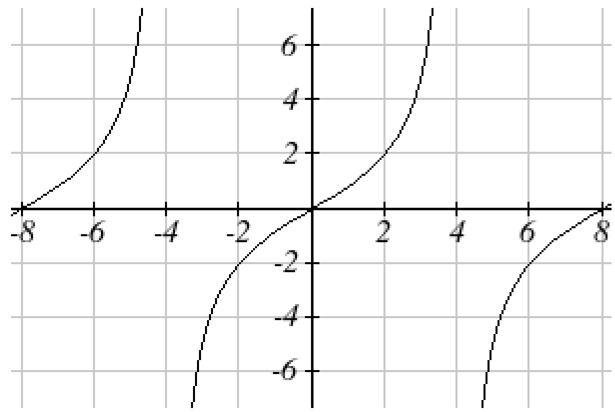
The **period** of the tangent function is π

The **domain** of the tangent function is $\theta \neq \frac{\pi}{2} + k\pi$, where k is an integer

The **range** of the tangent function is all real numbers, $(-\infty, \infty)$.

With the tangent function, like the sine and cosine functions, horizontal stretches/compressions are distinct from vertical stretches/compressions. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Example 12:



Find a formula for the function graphed here.

The graph has the shape of a tangent function, however the period appears to be 8. We can see one full continuous cycle from -4 to 4, suggesting a horizontal stretch. To stretch π to 8, the input values would have to be multiplied by $\frac{8}{\pi}$. Since the constant k in $f(\theta) = a\tan(k\theta)$ is the reciprocal of the horizontal stretch $\frac{8}{\pi}$, the equation must have form

$$f(\theta) = a\tan\left(\frac{\pi}{8}\theta\right)$$

We can also think of this the same way we did with sine and cosine. The period of the tangent function is π but it has been transformed and now it is 8; remember the ratio of the "normal period" to the "new period" is $\frac{\pi}{8}$ and so this becomes the value on the inside of the function that tells us how it was horizontally stretched.

To find the vertical stretch a, we can use a point on the graph. Using the point (2, 2)

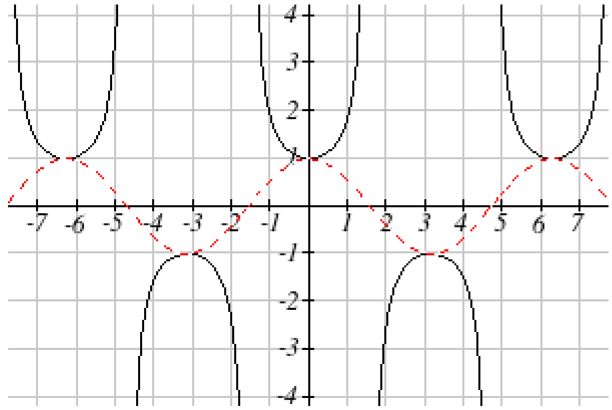
$$2 = a\tan\left(\frac{\pi}{8}\cdot 2\right) = a\tan\left(\frac{\pi}{4}\right) = a$$

So $a = 2$

This function would have a formula $f(\theta) = 2\tan\left(\frac{\pi}{8}\theta\right)$.

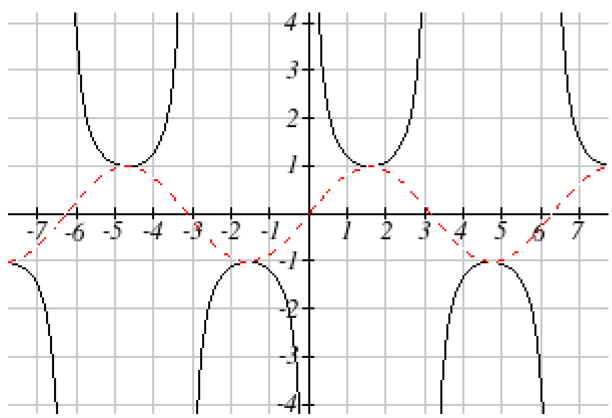
For the graph of secant, we remember the reciprocal identity where $\sec(\theta) = \frac{1}{\cos(\theta)}$. Notice that the function is undefined when the cosine is 0, leading to a vertical asymptote in the graph at $\frac{\pi}{2}, \frac{3\pi}{2}$, etc. Since the cosine is always no more than one in absolute value, the secant, being the reciprocal, will always be no less than one in absolute value. Using technology, we can generate the graph. The graph of the cosine is shown dashed so you can see the relationship.

$$f(\theta) = \sec(\theta) = \frac{1}{\cos(\theta)}$$



The graph of cosecant is similar. In fact, since $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, it follows that $\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$, suggesting the cosecant graph is a horizontal shift of the secant graph. This graph will be undefined where sine is 0. Recall from the unit circle that this occurs at $0, \pi, 2\pi$, etc. The graph of sine is shown dashed along with the graph of the cosecant.

$$f(\theta) = \csc(\theta) = \frac{1}{\sin(\theta)}$$



Features of the Graph of Secant and Cosecant

The secant and cosecant graphs have period 2π like the sine and cosine functions.

Secant has domain $\theta \neq \frac{\pi}{2} + k\pi$, where k is an integer

Cosecant has domain $\theta \neq k\pi$, where k is an integer

Both secant and cosecant have range of $(-\infty,-1] \cup [1,\infty)$

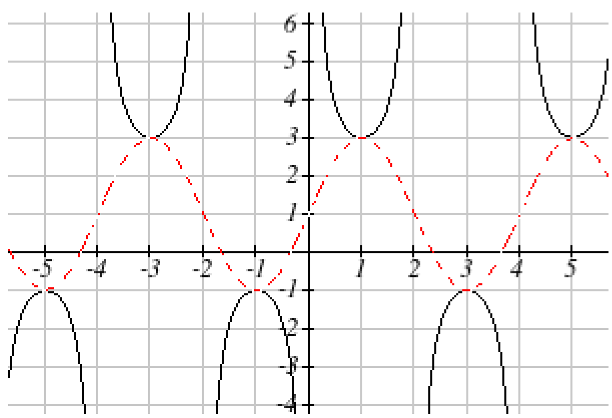
Example 13: Sketch a graph of $f(\theta) = 2\csc\left(\frac{\pi}{2}\theta\right) + 1$. What is the domain and range of this function?

The basic cosecant graph has vertical asymptotes at the integer multiples of π. Because of the factor $\frac{\pi}{2}$ inside the cosecant, the graph will be compressed by $\frac{2}{\pi}$, so the vertical asymptotes will be compressed to $\theta = \frac{2}{\pi} \cdot k\pi = 2k$. In other words, the graph will have vertical asymptotes at the integer multiples of 2, and the domain will correspondingly be $\theta \neq 2k$, where k is an integer.

The basic sine graph has a range of $[-1,1]$. The vertical stretch by 2 will stretch this to $[-2,2]$, and the vertical shift up 1 will shift the range of this function to $[-1,3]$.

The basic cosecant graph has a range of $(-\infty,-1] \cup [1,\infty)$. The vertical stretch by 2 will stretch this to $(-\infty,-2] \cup [2,\infty)$, and the vertical shift up 1 will shift the range of this function to $(-\infty,-1] \cup [3,\infty)$.

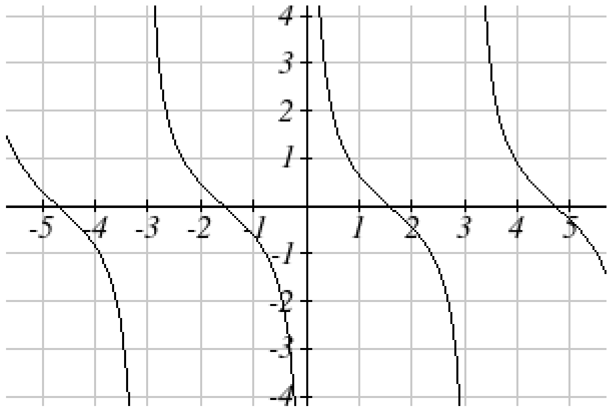
Sketching a graph,



Notice how the graph of the transformed cosecant relates to the graph of $f(\theta) = 2\sin\left(\frac{\pi}{2}\theta\right) + 1$ shown dashed.

Finally, we'll look at the graph of cotangent. Based on its definition as the ratio of cosine to sine, it will be undefined when the sine is zero: at $0, \pi, 2\pi$, etc. The resulting graph is similar to that of the tangent. In fact, it is a horizontal flip and shift of the tangent function, as we'll see shortly in Example 14.

$$f(\theta) = \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$



Features of the Graph of Cotangent

The cotangent graph has period π

Cotangent has domain $\theta \neq k\pi$, where k is an integer

Cotangent has range of all real numbers, $(-\infty, \infty)$

Earlier, we determined that the sine function was an odd function and the cosine was an even function by observing the graph and establishing the negative angle identities for cosine and sine. Similarly, you may notice from its graph that the tangent function appears to be odd. We can verify this using the negative angle identities for sine and cosine:

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

The secant, like the cosine it is based on, is an even function, while the cosecant, like the sine, is an odd function.

Negative Angle Identities Tangent, Cotangent, Secant and Cosecant

$$\tan(-\theta) = -\tan(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

$$\sec(-\theta) = \sec(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$

Example 14: Prove that $\tan(\theta) = -\cot\left(\theta - \frac{\pi}{2}\right)$

$\tan(\theta)$	Use the definition of tangent
$\frac{\sin(\theta)}{\cos(\theta)}$	Use the cofunction identity
$\frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)}$	Use the definition of cotangent
$\cot\left(\frac{\pi}{2} - \theta\right)$	Factor out negative
$\cot\left(-\left(\theta - \frac{\pi}{2}\right)\right)$	Use the negative angle identity
$-\cot\left(\theta - \frac{\pi}{2}\right)$	Final answer

5.6 Graphs of Trig Functions Practice

1. Sketch a graph of $f(x) = -3\sin(x)$.

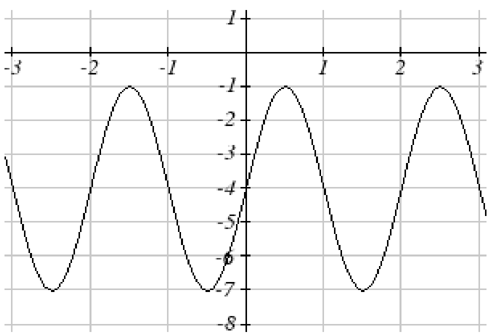
2. Sketch a graph of $f(x) = 4\sin(x)$.

3. Sketch a graph of $f(x) = 2\cos(x)$.

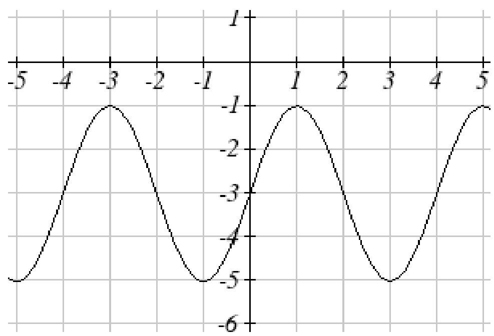
4. Sketch a graph of $f(x) = -4\cos(x)$.

For the graphs below, determine the amplitude, vertical shift, and period, then find a formula for the function.

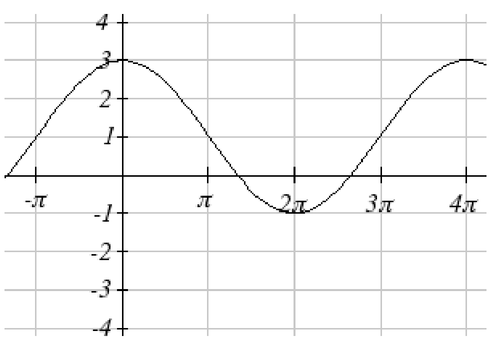
5.



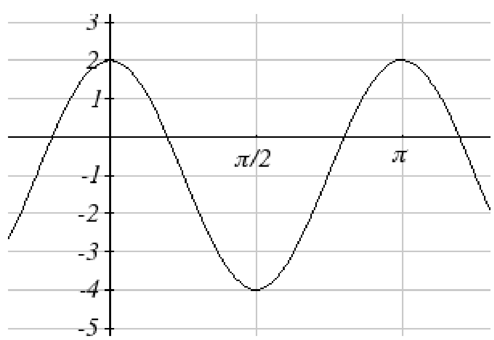
6.



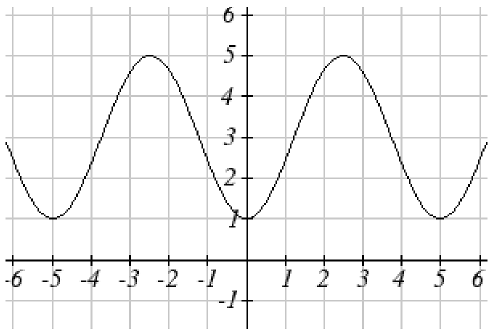
7.



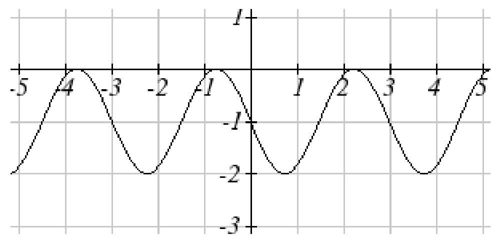
8.



9.



10.



For each of the following equations, find the amplitude, period, horizontal shift, and vertical shift.

11. $y = 3\sin\big(8(x+4)\big) + 8$

12. $y = 4\sin\left(\frac{\pi}{2}(x-3)\right) + 7$

13. $y = 2\sin(3x - 21) + 4$

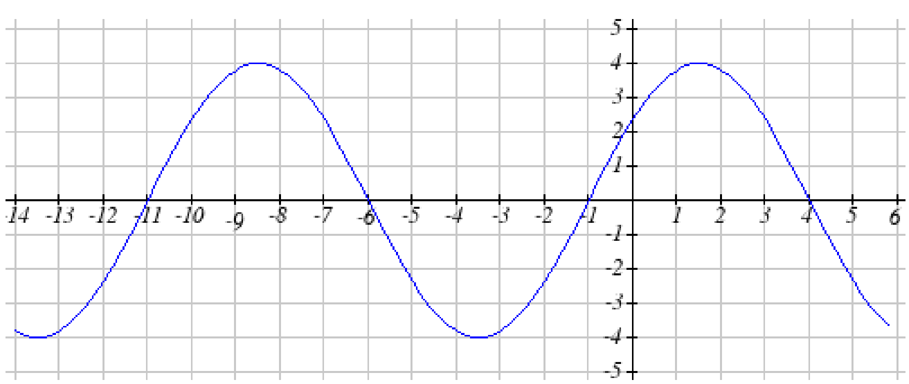
14. $y = 5\sin(5x + 20) - 2$

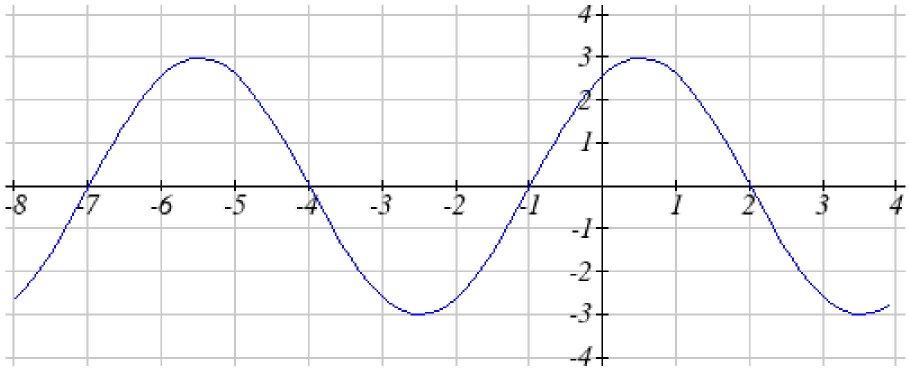
15. $y = \sin\left(\frac{\pi}{6}x + \pi\right)$

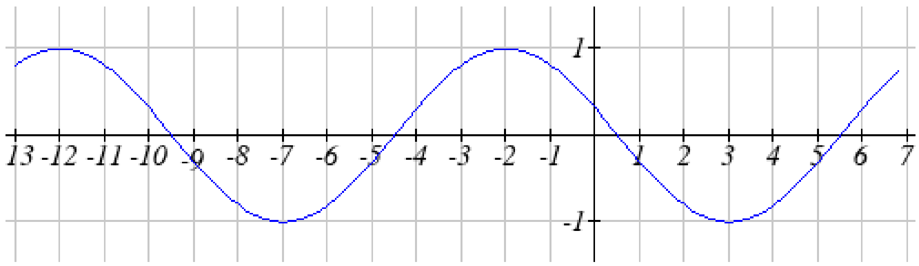
16. $y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$

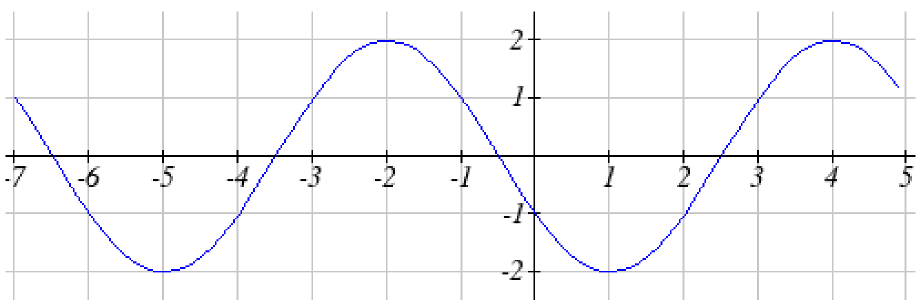
Find a formula for each of the functions graphed below.

17.



18. 

19. 

20. 

21. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 50 degrees at midnight and the high and low temperature during the day are 57 and 43 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

22. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68 degrees at midnight and the high and low temperature during the day are 80 and 56 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

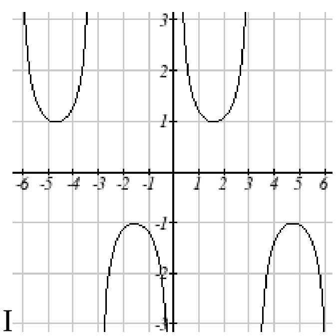
23. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives your height in meters above the ground t minutes after the wheel begins to turn.
   a. Find the amplitude, vertical shift, and period of $h(t)$.
   b. Find a formula for the height function $h(t)$.
   c. How high are you off the ground after 5 minutes?

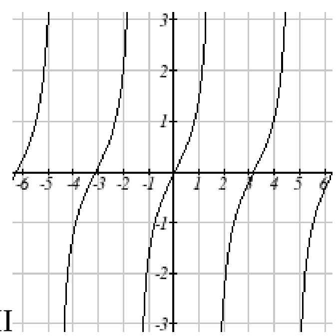
24. A Ferris wheel is 35 meters in diameter and boarded from a platform that is 3 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 8 minutes. The function $h(t)$ gives your height in meters above the ground t minutes after the wheel begins to turn.
   a. Find the amplitude, vertical shift, and period of $h(t)$.
   b. Find a formula for the height function $h(t)$.
   c. How high are you off the ground after 4 minutes?

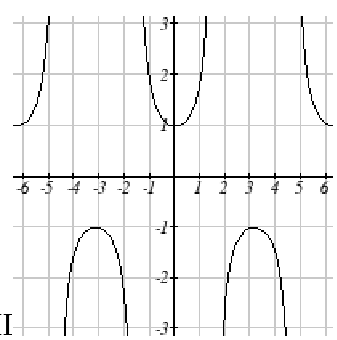
Match each trigonometric function with one of the graphs.

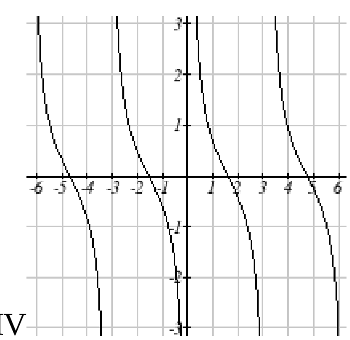
25. $f(x) = \tan(x)$ 26. $f(x) = \sec(x)$

27. $f(x) = \csc(x)$ 28. $f(x) = \cot(x)$

I


II


III


IV


Find the period and horizontal shift of each of the following functions.

29. $f(x) = 2\tan(4x - 32)$

30. $g(x) = 3\tan(6x + 42)$

31. $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

32. $k(x) = 3\sec\left(2\left(x + \frac{\pi}{2}\right)\right)$

33. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

34. $n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right)$

35. Sketch a graph of #31 above.

36. Sketch a graph of #32 above.

37. Sketch a graph of #33 above.

38. Sketch a graph of #34 above.

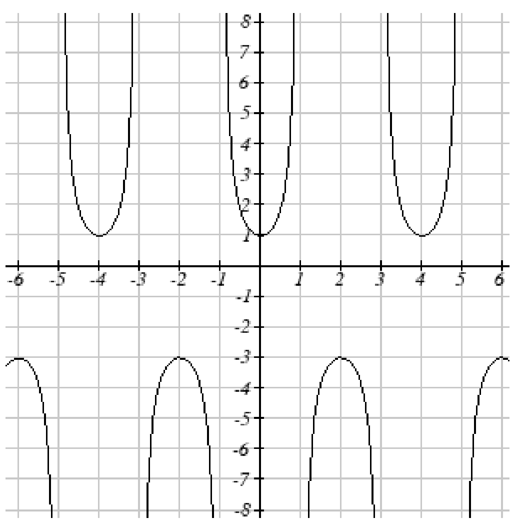
39. Sketch a graph of:

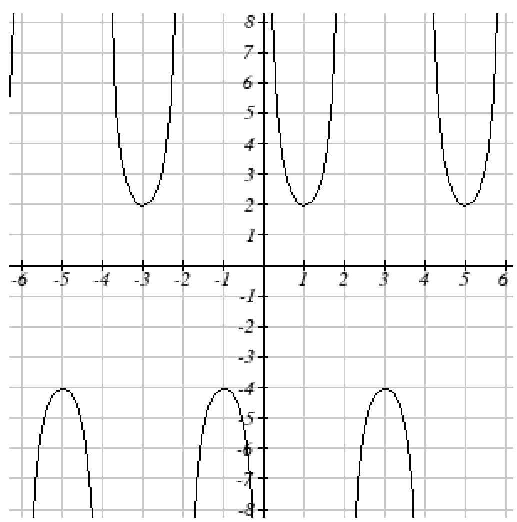
$$j(x) = \tan\left(\frac{\pi}{2}x\right)$$

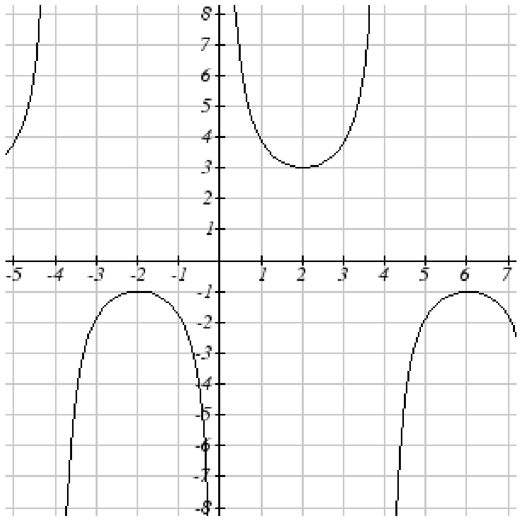
40. Sketch a graph of:

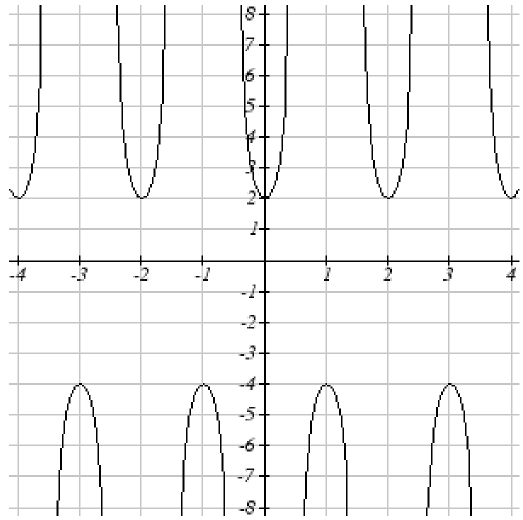
$$p(t) = 2\tan\left(t - \frac{\pi}{2}\right)$$

Find a formula for each function graphed below.

41. 

42. 

43. 

44. 

45. If $\tan x = -1.5$, find $\tan(-x)$

46. If $\tan x = 3$, find $\tan(-x)$

47. If $\sec x = 2$, find $\sec(-x)$

48. If $\sec x = -4$, find $\sec(-x)$

49. If $\csc x = -5$, find $\csc(-x)$

50. If $\csc x = 2$, find $\csc(-x)$

Simplify each of the following expressions completely.

51. $\cos(-x)\cot(-x) + \sin(-x)$

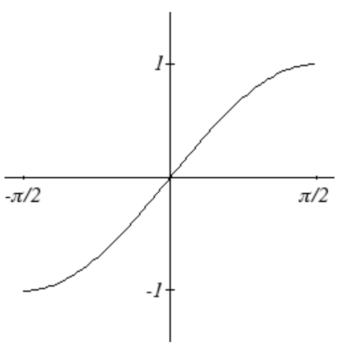
52. $\cos(-x) + \tan(-x)\sin(-x)$

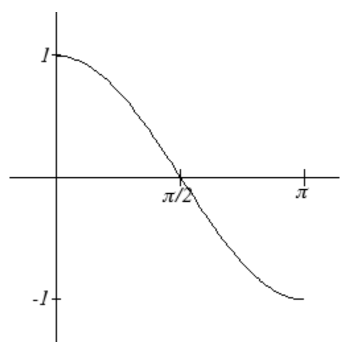
5.7 Inverse Trig Functions

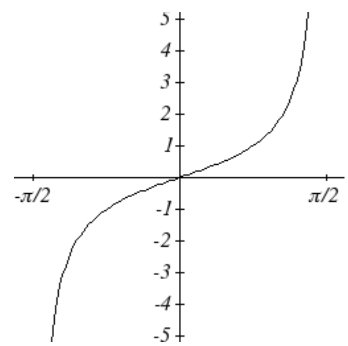
In previous sections we have evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

You probably are already recognizing an issue – that the sine, cosine, and tangent functions are not one-to-one functions. To define an inverse of these functions, we will need to restrict the domain of these functions to yield a new function that is one-to-one. We choose a domain for each function that includes the angle zero.

Sine, limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ Cosine, limited to $[0, \pi]$ Tangent, limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$







On these restricted domains, we can define the inverse sine, inverse cosine, and inverse tangent functions.

Inverse Sine, Cosine, and Tangent Functions

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin(\theta) = a$, then $\sin^{-1}(a) = \theta$

For angles in the interval $[0, \pi]$, if $\cos(\theta) = a$, then $\cos^{-1}(a) = \theta$

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan(\theta) = a$, then $\tan^{-1}(a) = \theta$

$\sin^{-1}(x)$ has domain $[-1,1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}(x)$ has domain $[-1,1]$ and range $[0, \pi]$

$\tan^{-1}(x)$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

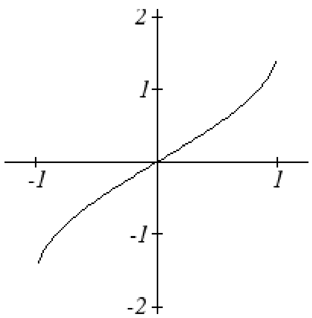
The $\sin^{-1}(x)$ is sometimes called the **arcsine** function, and notated $\arcsin(a)$.

The $\cos^{-1}(x)$ is sometimes called the **arccosine** function, and notated $\arccos(a)$.

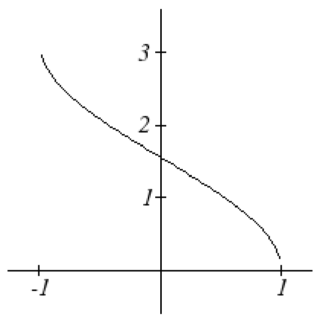
The $\tan^{-1}(x)$ is sometimes called the **arctangent** function, and notated $\arctan(a)$.

The graphs of the inverse functions are shown here:

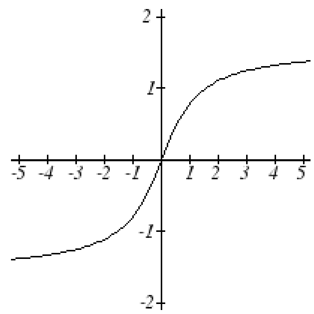
$\sin^{-1}(x)$



$\cos^{-1}(x)$



$\tan^{-1}(x)$



Notice that the output of each of these inverse functions is an *angle*.

Example 1: Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as asking what angle would have a sine value of $\frac{1}{2}$. In other words, what angle θ would satisfy $\sin(\theta) = \frac{1}{2}$? There are multiple angles that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be:

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Example 2: Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

We know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$.

$$\text{Sin}^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Example 3: Evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

We are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is:

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

Example 4: Evaluate $\tan^{-1}(1)$

We are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is:

$$\tan^{-1}(1) = \frac{\pi}{4}$$

Example 5: Evaluate $\sin^{-1}(0.97)$ using your calculator.

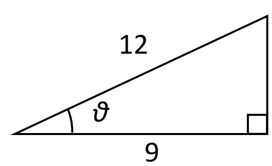
Since the output of the inverse function is an angle, your calculator will give you a degree value if in degree mode, and a radian value if in radian mode.

In radian mode, $\sin^{-1}(0.97) = 1.3252$

In degree mode, $\sin^{-1}(0.97) = 75.93°$

In Section 5.2, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trig functions, we can solve for the angles of a right triangle given two sides.

Example 6: Solve the triangle for the angle θ.



Since we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos(\theta) = \frac{9}{12}$$ Use the definition of the inverse

$$\cos^{-1}\left(\frac{9}{12}\right) = \theta$$ Evaluate

$\theta = 0.7227$ radians Final answer (radians and degrees)

$$\theta = 41.4096°$$

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can find exact values for the resulting expressions

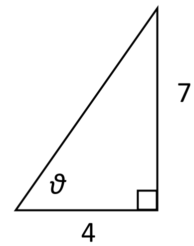
Example 7: Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$.

$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$	Evaluate the inside
$\cos\left(\frac{13\pi}{6}\right) = \frac{\sqrt{3}}{2}$	Evaluate the inverse sine of $\frac{\sqrt{3}}{2}$
$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$	Final answer

Example 8: Find an exact value for $\sin\left(cos^{-1}\left(\frac{4}{5}\right)\right)$.

$\sin\left(cos^{-1}\left(\frac{4}{5}\right)\right)$	Evaluate inside function
$\cos^{-1}\left(\frac{4}{5}\right)$	Rewrite
$\cos(\theta) = \frac{4}{5}$	Square
$\sin^2(\theta) + \left(\frac{4}{5}\right)^2 = 1$	Use Pythagorean identity
$\sin^2(\theta) + \frac{16}{25} = 1$	Subtract
$\sin^2(\theta) = \frac{9}{25}$	Square root
$\sin(\theta) = \pm\frac{3}{5}$	Inverse cosine is positive, so sine must be positive
$\frac{3}{5}$	Final answer

Example 9: Find an exact value for $\sin\left(tan^{-1}\left(\frac{7}{4}\right)\right)$.



While we could use a similar technique as in the last example, we will demonstrate a different technique here. From the inside, we know there is an angle so $\tan(\theta) = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle. Using the Pythagorean Theorem:

$4^2 + 7^2 = \text{hypotenuse}^2$	Solve for missing side
$\sqrt{65}$	Evaluate the sine
$\sin(\theta) = \frac{7}{\sqrt{65}} = \frac{7\sqrt{65}}{65}$	Final answer

We can also find compositions involving algebraic expressions.

Example 10: Find a simplified expression for $\cos\left(sin^{-1}\left(\frac{x}{3}\right)\right)$, for $-3 \leq x \leq 3$.

$\cos\left(sin^{-1}\left(\frac{x}{3}\right)\right)$	Evaluate inside function
$\sin^{-1}\left(\frac{x}{3}\right)$	Rewrite
$\sin(\theta) = \frac{x}{3}$	Square
$\left(\frac{x}{3}\right)^2 + \cos^2 x = 1$	Use Pythagorean identity
$\frac{x^2}{9} + \cos^2(x) = 1$	Subtract
$\cos^2(x) = \frac{9 - x^2}{9}$	Square root
$\cos(\theta) = \pm\frac{\sqrt{9 - x^2}}{3}$	Inverse sine is on interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so cosine must be positive
$\frac{\sqrt{9 - x^2}}{3}$	Final answer

5.7 Inverse Trig Functions Practice

Evaluate the following expressions

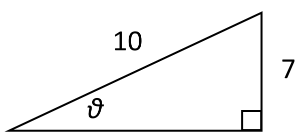
1. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
3. $\sin^{-1}\left(-\frac{1}{2}\right)$
4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
5. $\cos^{-1}\left(\frac{1}{2}\right)$
6. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
7. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
8. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
9. $\tan^{-1}(1)$
10. $\tan^{-1}(\sqrt{3})$
11. $\tan^{-1}(-\sqrt{3})$
12. $\tan^{-1}(-1)$

Use your calculator to evaluate each expression

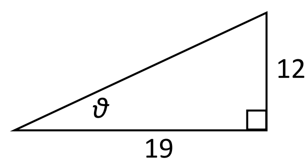
13. $\cos^{-1}(-0.4)$
14. $\cos^{-1}(0.8)$
15. $\sin^{-1}(-0.8)$
16. $\tan^{-1}(6)$

Find the angle θ

17.



18.



Evaluate the following expressions:

19. $\sin^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)$
20. $\cos^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$
21. $\sin^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$
22. $\cos^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$
23. $\cos\left(sin^{-1}\left(\frac{3}{7}\right)\right)$
24. $\sin\left(cos^{-1}\left(\frac{4}{9}\right)\right)$
25. $\cos(\tan^{-1}(4))$
26. $\tan\left(sin^{-1}\left(\frac{1}{3}\right)\right)$

Find a simplified expression for each of the following:

27. $\sin\left(cos^{-1}\left(\frac{x}{5}\right)\right)$, for $-5 \le x \le 5$
28. $\tan\left(cos^{-1}\left(\frac{x}{2}\right)\right)$, for $-2 \le x \le 2$
29. $\sin(\tan^{-1}(3x))$
30. $\cos(\tan^{-1}(4x))$

Chapter 6

Analytic Trigonometry

6.1 Solving Trigonometric Equations

In Section 5.6, we determined the height of a rider on the London Eye Ferris wheel could be determined by the equation $h(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$.
If we wanted to know length of time during which the rider is more than 100 meters above ground, we would need to solve equations involving trig functions.

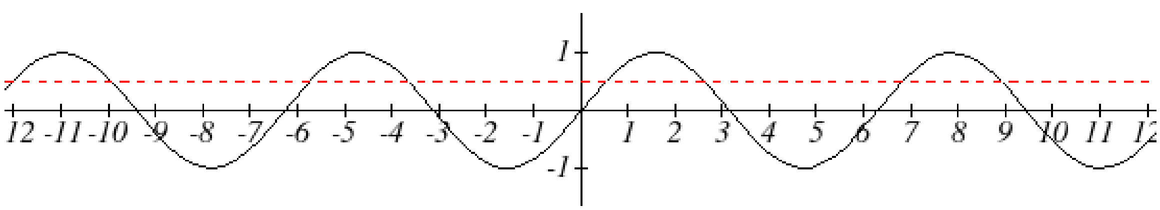
Solving using known values

In the last chapter, we learned sine and cosine values at commonly encountered angles. We can use these to solve sine and cosine equations involving these common angles.

Example 1: Solve $\sin(t) = \frac{1}{2}$ for all possible values of t.

Notice this is asking us to identify all angles, t, that have a sine value of $t = \frac{1}{2}$. While evaluating a function always produces one result, solving for an input can yield multiple solutions. Two solutions should immediately jump to mind from the last chapter: $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ because they are the common angles on the unit circle.

Looking at a graph confirms that there are more than these two solutions. While eight are seen on this graph, there are an infinite number of solutions!

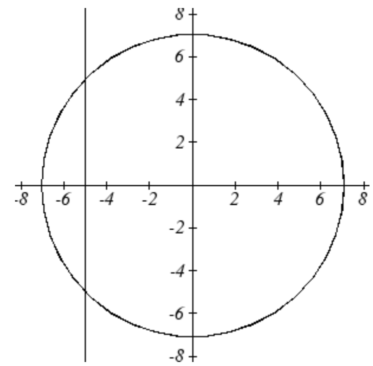


Remember that any coterminal angle will also have the same sine value, so any angle coterminal with these two is also a solution. Coterminal angles can be found by adding full rotations of 2π, so we end up with a set of solutions:

$t = \frac{\pi}{6} + 2\pi k$ where k is an integer, and $t = \frac{5\pi}{6} + 2\pi k$ where k is an integer

Example 2: A circle of radius $5\sqrt{2}$ intersects the line $x = -5$ at two points. Find the angles θ on the interval $0 \le \theta < 2\pi$, where the circle and line intersect.

The x coordinate of a point on a circle can be found as $x = rcos(\theta)$, so the x coordinate of points on this circle would be $x = 5\sqrt{2}\cos(\theta)$. To find where the line $x = -5$ intersects the circle, we can solve for where the x value on the circle would be -5



$$-5 = 5\sqrt{2}\cos(\theta) \qquad \text{Divide by } 5\sqrt{2}$$

$$-\frac{1}{\sqrt{2}} = \cos(\theta) \qquad \text{Rationalize denominator}$$

$$-\frac{\sqrt{2}}{2} = \cos(\theta)$$

We can recognize this as one of our special cosine values from our unit circle, and it corresponds with angles $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{4}$

Example 3: The depth of water at a dock rises and falls with the tide, following the equation $f(t) = 4\sin\left(\frac{\pi}{12}t\right) + 7$, where t is measured in hours after midnight. A boat requires a depth of 9 feet to tie up at the dock. Between what times will the depth be 9 feet?

To find when the depth is 9 feet, we need to solve *f(t)* = 9.

$$4\sin\left(\frac{\pi}{12}t\right) + 7 = 9 \qquad \text{Subtract 7}$$

$$4\sin\left(\frac{\pi}{12}t\right) = 2 \qquad \text{Divide by 4}$$

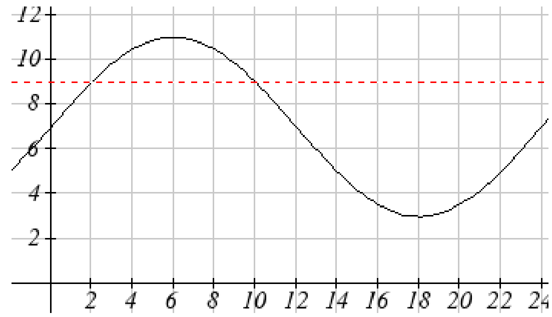
$$\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2} \qquad \text{Sine is } \tfrac{1}{2} \text{ at } \tfrac{\pi}{6} + 2\pi k \text{ and } \tfrac{5\pi}{6} + 2\pi k$$

$$\frac{\pi}{12}t = \frac{\pi}{6} + 2\pi k \quad \text{or} \quad \frac{\pi}{12}t = \frac{5\pi}{6} + 2\pi k \qquad \text{Solve by multiplying by } \tfrac{12}{\pi}$$

$$t = 2 + 24k \quad \text{or} \quad t = 10 + 24k \qquad \text{This is 2 hours and 10 hours past midnight}$$

Between 2:00 am and 10:00am   Final answer

Notice how in both scenarios, the $24k$ shows how every 24 hours the cycle will be repeated.



In the previous example, looking back at the original simplified equation $\sin\left(\frac{\pi}{12}t\right) = \frac{1}{2}$, we can use the ratio of the "normal period" to the stretch factor to find the period. $\frac{2\pi}{\frac{\pi}{12}} = 2\pi\left(\frac{12}{\pi}\right) = 24$; notice that the sine function has a period of 24, which is reflected in the solutions: there were two unique solutions on one full cycle of the sine function, and additional solutions were found by adding multiples of a full period.

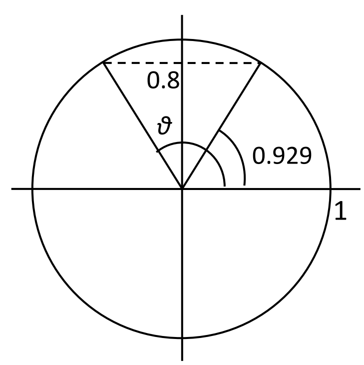
Example 4: Use the inverse sine function to find one solution to $\sin(\theta) = 0.8$.

Since this is not a known unit circle value, calculating the inverse, $\theta = \sin^{-1}(0.8)$. This requires a calculator and we must approximate a value for this angle. If your calculator is in degree mode, your calculator will give you an angle in degrees as the output. If your calculator is in radian mode, your calculator will give you an angle in radians. In radians, $\theta = \sin^{-1}(0.8) = 0.927$, or in degrees, $\theta = \sin^{-1}(0.8) = 53.130°$.

If you are working with a composed trig function and you are not solving for an angle, you will want to ensure that you are working in radians. In calculus, we will almost always want to work with radians since they are unit-less.

Notice that the inverse trig functions do exactly what you would expect of any function – for each input they give exactly one output. While this is necessary for these to be a function, it means that to find *all* the solutions to an equation like $\sin(\theta) = 0.8$, we need to do more than just evaluate the inverse function.

Example 5: Find all solutions to $\sin(\theta) = 0.8$.



We would expect two unique angles on one cycle to have this sine value. In the previous example, we found one solution to be $\theta = \sin^{-1}(0.8) = 0.927$. To find the other, we need to answer the question "what other angle has the same sine value as an angle of 0.927?" On a unit circle, we would recognize that the second angle would have the same reference angle and reside in the second quadrant. This second angle would be located at $\theta = \pi - \sin^{-1}(0.8)$, or approximately $\theta = \pi - 0.927 = 2.214$

To find more solutions we recall that angles coterminal with these two would have the same sine value, so we can add full cycles of 2π.

$\theta = \sin^{-1}(0.8) + 2\pi k$ and $\theta = \pi - \sin^{-1}(0.8) + 2\pi k$ where k is an integer,

or approximately, $\theta = 0.927 + 2\pi k$ and $\theta = 2.214 + 2\pi k$ where k is an integer.

Example 6: Find all solutions to $\sin(x) = -\frac{8}{9}$ on the interval $0° \le x < 360$.

First we will turn our calculator to degree mode. Using the inverse, we can find one solution $x = \sin^{-1}\left(-\frac{8}{9}\right) = -62.734°$. While this angle satisfies the equation, it does not lie in the domain we are looking for. To find the angles in the desired domain, we start looking for additional solutions.

First, an angle coterminal with $-62.734°$will have the same sine. By adding a full rotation, we can find an angle in the desired domain with the same sine.

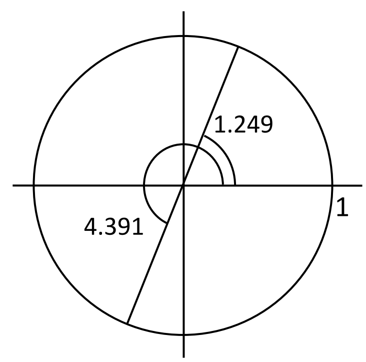
$$x = -62.734° + 360° = 297.266°$$

There is a second angle in the desired domain that lies in the third quadrant. Notice that 62.734º is the reference angle for all solutions, so this second solution would be 62.734º past 180º

$$x = 62.734° + 180 = 242.734°$$

The two solutions on $0° \le x < 360°$ are $x = 297.266°$ and $x = 242.734°$

Example 7: Find all solutions to $\tan(x) = 3$ on $0 \le x < 2\pi$.



Using the inverse tangent function, we can find one solution $x = \tan^{-1}(3) = 1.249$. Unlike the sine and cosine, the tangent function only attains any output value once per cycle, so there is no second solution in any one cycle.

By adding π, a full period of tangent function, we can find a second angle with the same tangent value. If additional solutions were desired, we could continue to add multiples of π, so all solutions would take on the form $x = 1.249 + k\pi$, however we are only interested in $0 \le x < 2\pi$.

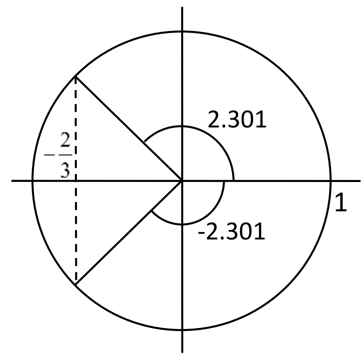
$$x = 1.249 + \pi = 4.391$$

The two solutions on $0 \le x < 2\pi$ are $x = 1.249$ and $x = 4.391$.

Example 8: Solve $3\cos(t) + 4 = 2$ for all solutions on one cycle, $0 \le t < 2\pi$

$3\cos(t) + 4 = 2$	Isolate the cosine, subtract 4
$3\cos(t) = -2$	Divide by 3
$\cos(t) = -\frac{2}{3}$	Calculate inverse
$\cos^{-1}\left(-\frac{2}{3}\right) = 2.301$	

Thinking back to the circle, a second angle with the same cosine would be located in the third quadrant. Notice that the location of this angle could be represented as $t = -2.301$. To represent this as a positive angle we could find a coterminal angle by adding a full cycle.



$$t = -2.301 + 2\pi = 3.982$$

The equation has two solutions between 0 and 2π, at $t = 2.301$ and $t = 3.982$.

Example 9: Solve $\cos(3t) = 0.2$ for all solutions on two cycles, $0 \le t < \frac{4\pi}{3}$.

As before, with a horizontal compression it can be helpful to make a substitution, $u = 3t$. Making this substitution simplifies the equation to a form we have already solved.

$\cos(u) = 0.2$	Inverse cosine
$u = \cos^{-1}(0.2) = 1.369$	Second solution in 4th quadrant
$u = 2\pi - 1.369 = 4.914$	Need two cycles, add 2π to each solution
$u = 1.369 + 2\pi = 7.653$ $u = 4.914 + 2\pi = 11.197$	Undo the substitution
$3t = 1.369, 4.914, 7.653, 11.197$	Divide each solution by 3
$t = 0.456, 1.638, 2.551, 3.732$	Final answer

Example 10: Solve $3\sin(\pi t) = -2$ for all solutions.

$3\sin(\pi t) = -2$	Divide by 3
$\sin(\pi t) = -\frac{2}{3}$	Substitution $u = \pi t$
$\sin(u) = -\frac{2}{3}$	Inverse sine
$u = \sin^{-1}\left(-\frac{2}{3}\right) = -0.730$	Second angle in third quadrant
$u = \pi + 0.730 = 3.871$	All solution by adding $2\pi k$ where k is an integer
$u = -0.730 + 2\pi k$ $u = 3.871 + 2\pi k$	Undo substitution
$\pi t = -0.730 + 2\pi k$ $\pi t = 3.871 + 2\pi k$	Divide each solution by π
$t = -0.232 + 2k$ $t = 1.232 + 2k$	Final answer

Solving Trig Equations

1. Isolate the trig function on one side of the equation
2. Make a substitution for the inside of the sine, cosine, or tangent (or other trig function)
3. Use inverse trig functions to find one solution
4. Use symmetries to find a second solution on one cycle (when a second exists)
5. Find additional solutions if needed by adding full periods
6. Undo the substitution

We now can return to the question we began the section with.

Example 11: The height of a rider on the London Eye Ferris wheel can be determined by the equation $h(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$. How long is the rider more than 100 meters above ground?

To find how long the rider is above 100 meters, we first find the times at which the rider is at a height of 100 meters by solving $h(t) = 100$.

$$100 = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$ Isolate the cosine, subtract 69.5

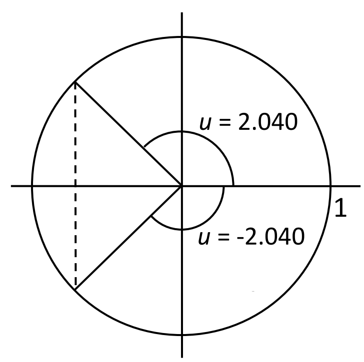
$$30.5 = -67.5\cos\left(\frac{\pi}{15}t\right)$$ Divide by -67.5

$$\frac{30.5}{-67.5} = \cos\left(\frac{\pi}{15}t\right)$$ Substitution

$$\frac{30.5}{-67.5} = \cos(u)$$ Inverse cosine

$$u = \cos^{-1}\left(\frac{30.5}{-67.5}\right) = 2.040$$ Use unit circle

This angle is in the second quadrant. A second angle with the same cosine would be symmetric in the third quadrant. This angle could be represented as $u = -2.040$, but we need a coterminal positive angle, so we add 2π:



$$u = 2\pi - 2.040 = 4.244$$

Now we can undo the substitution to solve for t

$\frac{\pi}{15}t = 2.040$, so $t = 9.74$ minutes after the start of the ride

$\frac{\pi}{15}t = 4.244$ so $t = 20.264$ minutes after the start of the ride

A rider will be at 100 meters after 9.740 minutes, and again after 20.264. From the behavior of the height graph, we know the rider will be above 100 meters between these times. A rider will be above 100 meters for $20.265 - 9.740 = 10.523$ minutes of the ride.

6.1 Solving Trigonometric Equations Practice

Find all solutions on the interval $0 \leq \theta < 2\pi$

1. $2\sin(\theta) = -\sqrt{2}$
2. $2\sin(\theta) = \sqrt{3}$
3. $2\cos(\theta) = 1$
4. $2\cos(\theta) = -\sqrt{2}$
5. $\sin(\theta) = 1$
6. $\sin(\theta) = 0$
7. $\cos(\theta) = 0$
8. $\cos(\theta) = -1$

Find all solutions

9. $2\cos(\theta) = \sqrt{2}$
10. $2\cos(\theta) = -1$
11. $2\sin(\theta) = -1$
12. $2\sin(\theta) = -\sqrt{3}$

Find all solutions

13. $2\sin(3\theta) = 1$
14. $2\sin(2\theta) = \sqrt{3}$
15. $2\sin(3\theta) = -\sqrt{2}$
16. $2\sin(3\theta) = -1$
17. $2\cos(2\theta) = 1$
18. $2\cos(2\theta) = \sqrt{3}$
19. $2\cos(3\theta) = -\sqrt{2}$
20. $2\cos(2\theta) = -1$
21. $\cos\left(\frac{\pi}{4}\theta\right) = -1$
22. $\sin\left(\frac{\pi}{3}\theta\right) = -1$
23. $2\sin(\pi\theta) = 1$
24. $2\cos\left(\frac{\pi}{5}\theta\right) = \sqrt{3}$

Find all solutions on the interval $0 \leq x < 2\pi$

25. $\sin(x) = 0.27$
26. $\sin(x) = 0.48$
27. $\sin(x) = -0.58$
28. $\sin(x) = -0.34$
29. $\cos(x) = -0.55$
30. $\sin(x) = 0.28$
31. $\cos(x) = 0.71$
32. $\cos(x) = -0.07$

Find the first two positive solutions.

33. $7\sin(6x) = 2$
34. $7\sin(5x) = 6$
35. $5\cos(3x) = -3$
36. $3\cos(4x) = 2$
37. $3\sin\left(\frac{\pi}{4}x\right) = 2$
38. $7\sin\left(\frac{\pi}{5}x\right) = 6$
39. $5\cos\left(\frac{\pi}{3}x\right) = 1$
40. $3\cos\left(\frac{\pi}{2}x\right) = -2$

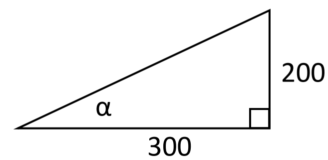
6.2 Modeling with Trigonometric Functions

Solving right triangles for angles

In Section 5.2, we used trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trig functions, we can solve for the angles of a right triangle given two sides.

Example 1: An airplane needs to fly to an airfield located 300 miles east and 200 miles north of its current location. At what heading should the airplane fly? In other words, if we ignore air resistance or wind speed, how many degrees north of east should the airplane fly?

We might begin by drawing a picture and labeling all of the known information. Drawing a triangle, we see we are looking for the angle α. In this triangle, the side opposite the angle α is 200 miles and the side adjacent is 300 miles. Since we know the values for the opposite and adjacent sides, it makes sense to use the tangent function.

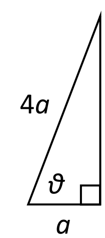


$$\tan(\alpha) = \frac{200}{300}$$ Inverse

$$\alpha = \tan^{-1}\left(\frac{200}{300}\right) = 0.588 \quad \text{or} \quad 33.7°$$ Final answer

The airplane needs to fly at a heading of 33.7 degrees north of east.

Example 2: OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle such a ladder forms with the ground.



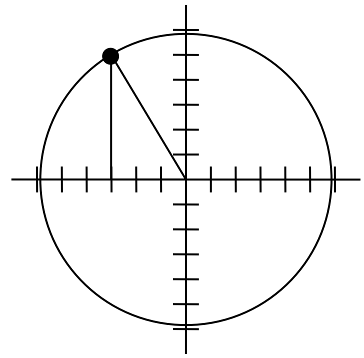
For any length of ladder, the base needs to be one quarter of the distance the foot of the ladder is away from the wall. Equivalently, if the base is a feet from the wall, the ladder can be $4a$ feet long. Since a is the side adjacent to the angle and $4a$ is the hypotenuse, we use the cosine function.

$$\cos(\theta) = \frac{a}{4a} = \frac{1}{4}$$ Inverse

$$\theta = \cos^{-1}\left(\frac{1}{4}\right) = 75.52°$$ Final answer

The ladder forms a 75.52 degree angle with the ground.

Example 3: In a video game design, a map shows the location of other characters relative to the player, who is situated at the origin, and the direction they are facing. A character currently shows on the map at coordinates $(-3, 5)$. If the player rotates counterclockwise by 20 degrees, then the objects in the map will correspondingly rotate 20 degrees clockwise. Find the new coordinates of the character.



To rotate the position of the character, we can imagine it as a point on a circle, and we will change the angle of the point by 20 degrees. To do so, we first need to find the radius of this circle and the original angle.

Drawing a right triangle inside the circle, we can find the radius using the Pythagorean Theorem:

$$(-3)^2 + 5^2 = r^2$$

$$r = \sqrt{9 + 25} = \sqrt{34}$$

To find the angle, we need to decide first if we are going to find the acute angle of the triangle, the reference angle, or if we are going to find the angle measured in standard position. While either approach will work, in this case we will do the latter. Since for any point on a circle we know $x = r\cos(\theta)$, using our given information we get

$$-3 = \sqrt{34}\cos(\theta) \qquad \text{Divide by } \sqrt{34}$$

$$-\frac{3}{\sqrt{34}} = \cos(\theta) \qquad \text{Inverse}$$

$$\theta = \cos^{-1}\left(-\frac{3}{\sqrt{34}}\right) = 120.964° \qquad \text{Second quadrant as desired}$$

Rotating the point clockwise by 20 degrees, the angle of the point will decrease to 100.964 degrees. We can then evaluate the coordinates of the rotated point

$$x = \sqrt{34}\cos(100.964°) = -1.109$$

$$y = \sqrt{34}\sin(100.964°) = 5.725$$

The coordinates of the character on the rotated map will be $(-1.109, 5.725)$.

Modeling with sinusoidal functions

Many modeling situations involve functions that are periodic. Previously we learned that sinusoidal functions are a special type of periodic function. Problems that involve quantities that oscillate can often be modeled by a sine or cosine function and once we create a suitable model for the problem we can use that model to answer various questions.

Example 4: The hours of daylight in Seattle oscillate from a low of 8.5 hours in January to a high of 16 hours in July. When should you plant a garden if you want to do it during a month where there are 14 hours of daylight?

To model this, we first note that the hours of daylight oscillate with a period of 12 months. With a low of 8.5 and a high of 16, the vertical shift will be halfway between these values, at $\frac{16+8.5}{2} = 12.25$. The amplitude will be half the difference between the highest and lowest values: $\frac{16-8.5}{2} = 3.75$, or equivalently the distance from the vertical shift to the high or low value, $16 - 12.25 = 3.75$. Letting January be $t = 0$, the graph starts at the lowest value, so it can be modeled as a flipped cosine graph. Putting this together, we get a model:

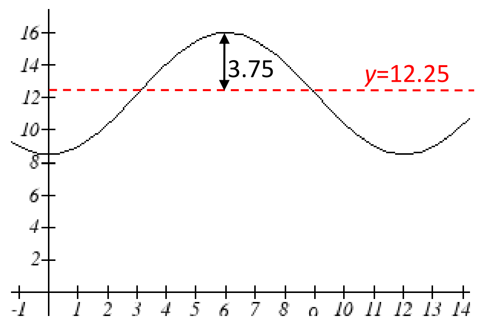
$$h(t) = -3.75\cos\left(\frac{\pi}{6}t\right) + 12.25$$

$-cos(t)$ represents the flipped cosine,

3.75 is the amplitude,

12.25 is the vertical shift,

$\frac{2\pi}{12} = \frac{\pi}{6}$ corresponds to the horizontal stretch, found by using the ratio of the “original period / new period”



$h(t)$ is our model for hours of day light t months after January.

To find when there will be 14 hours of daylight, we solve $h(t) = 14$

$$14 = -3.75cos\left(\frac{\pi}{6}t\right) + 12.25$$ Subtract 12.5

$$1.75 = -3.75\cos\left(\frac{\pi}{6}t\right)$$ Divide -3.75

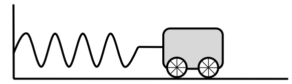
$$-\frac{1.75}{3.75} = \cos\left(\frac{\pi}{6}t\right)$$ Inverse cosine

$$\frac{\pi}{6}t = \cos^{-1}\left(-\frac{1.75}{3.75}\right) = 2.0563$$ Multiply by reciprocal

$$t = 2.0563 \cdot \frac{6}{\pi} = 3.927 \text{ months}$$ Final answer

While there would be a second time in the year when there are 14 hours of daylight, since we are planting a garden, we would want to know the first solution, in spring, so we do not need to find the second solution in this case.

Example 5: An object is connected to the wall with a spring that has a natural length of 20 cm. The object is pulled back 8 cm past the natural length and released. The object oscillates 3 times per second. Find an equation for the horizontal position of the object ignoring the effects of friction. How much time during each cycle is the object more than 27 cm from the wall?



If we use the distance from the wall, x, as the desired output, then the object will oscillate equally on either side of the spring's natural length of 20, putting the vertical shift of the function at 20 cm.

If we release the object 8 cm past the natural length, the amplitude of the oscillation will be 8 cm.

We are beginning at the largest value and so this function can most easily be modeled using a cosine function.

Since the object oscillates 3 times per second, it has a frequency of 3 and the period of one oscillation is $\frac{1}{3}$ of second. Using this we find the horizontal compression using the ratios of the periods: $\frac{2\pi}{\frac{1}{3}} = 6\pi$.

Using all this, we can build our model:

$$x(t) = 8\cos(6\pi t) + 20$$

To find when the object is 27 cm from the wall, we can solve $x(t) = 27$

$$27 = 8\cos(6\pi t) + 20 \qquad \text{Subtract 20}$$

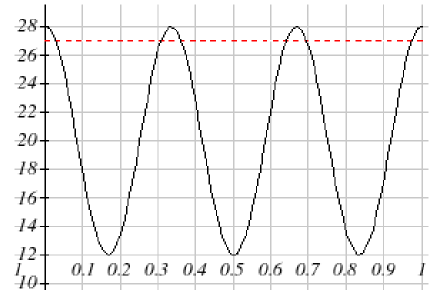
$$7 = 8\cos(6\pi t) \qquad \text{Divide by 8}$$

$$\frac{7}{8} = \cos(6\pi t) \qquad \text{Inverse}$$

$$6\pi t = \cos^{-1}\left(\frac{7}{8}\right) = 0.505 \qquad \text{Divide by } 6\pi$$

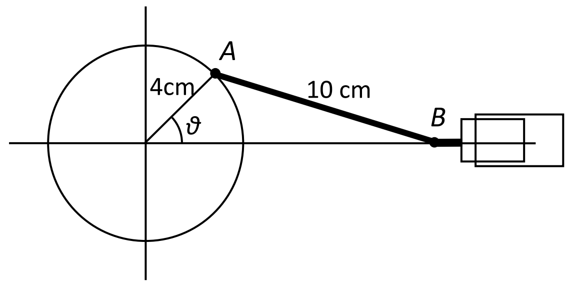
$$t = \frac{0.505}{6\pi} = 0.0268 \qquad \text{Consider the graph}$$

Based on the shape of the graph, we can conclude that the object will spend the first 0.0268 seconds more than 27 cm from the wall. Based on the symmetry of the function, the object will spend another 0.0268 seconds more than 27 cm from the wall at the end of the cycle. Altogether, the object spends 0.0536 seconds each cycle at a distance greater than 27 cm from the wall.



In some problems, we can use trigonometric functions to model behaviors more complicated than the basic sinusoidal function.

Example 6: A rigid rod with length 10 cm is attached to a circle of radius 4cm at point A as shown here. The point B is able to freely move along the horizontal axis, driving a piston[1]. If the wheel rotates counterclockwise at 5 revolutions per second, find the location of point B as a function of time. When will the point B be 12 cm from the center of the circle?



To find the position of point B, we can begin by finding the coordinates of point A. Since it is a point on a circle with radius 4, we can express its coordinates as $(4\cos(\theta), 4\sin(\theta))$, where θ is the angle shown.

The angular velocity is 5 revolutions per second, or equivalently 10π radians per second. After t seconds, the wheel will rotate by $\theta = 10\pi t$ radians. Substituting this, we can find the coordinates of A in terms of t.

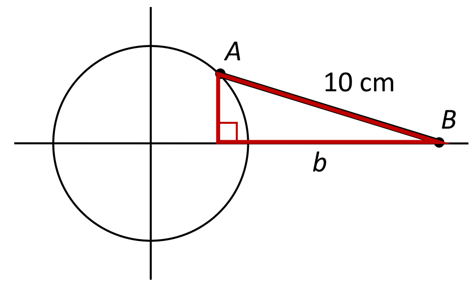
$$(4\cos(10\pi t), 4\sin(10\pi t))$$

Notice that this is the same value we would have obtained by observing that the period of the rotation is $\frac{1}{5}$ of a second and calculating the stretch/compression factor:

$$\frac{2\pi}{\frac{1}{5}} = 10\pi$$

[1] For an animation of this situation, see http://www.mathdemos.org/mathdemos/sinusoidapp/engine1.gif

Now that we have the coordinates of the point A, we can relate this to the point B. By drawing a vertical line segment from A to the horizontal axis, we can form a right triangle. The height of the triangle is the y coordinate of the point A: $4\sin(10\pi t)$. Using the Pythagorean Theorem, we can find the base length of the triangle:



$$(4\sin(10\pi t))^2 + b^2 = 10^2 \qquad \text{Solve for } b^2$$

$$b^2 = 100 - 16\sin^2(10\pi t) \qquad \text{Square root}$$

$$b = \sqrt{100 - 16\sin^2(10\pi t)}$$

Looking at the x coordinate of the point A, we can see that the triangle we drew is shifted to the right of the y axis by $4\cos(10\pi t)$. Combining this offset with the length of the base of the triangle gives the x coordinate of the point B:

$$x(t) = 4\cos(10\pi t) + \sqrt{100 - 16\sin^2(10\pi t)}$$

To solve for when the point B will be 12 cm from the center of the circle, we need to solve $x(t) = 12$.

$$12 = 4\cos(10\pi t) + \sqrt{100 - 16\sin^2(10\pi t)} \qquad \text{Isolate the square root}$$

$$12 - 4\cos(10\pi t) = \sqrt{100 - 16\sin^2(10\pi t)} \qquad \text{Square both sides}$$

$$144 - 96\cos(10\pi t) + 16\cos^2(10\pi t) = 100 - 16\sin^2(10\pi t) \qquad \text{Move all terms to the left}$$

$$44 - 96\cos(10\pi t) + 16\sin^2(10\pi t) + 16\cos^2(10\pi t) = 0 \qquad \text{Factor out 16}$$

$$44 - 96\cos(10\pi t) + 16(\sin^2(10\pi t) + \cos^2(10\pi t)) = 0 \qquad \text{Simplify: } \sin^2(\theta) + \cos^2(\theta) = 1$$

$$44 - 96\cos(10\pi t) + 16 = 0 \qquad \text{Combine like terms}$$

$$-96\cos(10\pi t) + 60 = 0 \qquad \text{Subtract 60}$$

$$-96\cos(10\pi t) = -60 \qquad \text{Divide } -96$$

$$\cos(10\pi t) = \frac{60}{96} \qquad \text{Substitution}$$

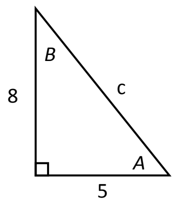
$\cos(u) = \frac{60}{96}$	Inverse cosine
$u = \cos^{-1}\left(\frac{60}{96}\right) = 0.896$	Second solution by symmetry
$u = 2\pi - 0.896 = 5.388$	Undo substitution
$10\pi t = 0.896, 5.388$	Divide each by 10π
$t = 0.0285, 0.1715$ seconds	Final answer

The point B will be 12 cm from the center of the circle 0.0285 seconds after the process begins, 0.1715 seconds after the process begins, and every $\frac{1}{5}$ of a second after each of those values.

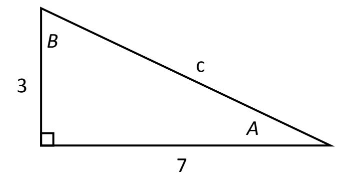
6.2 Modeling with Trigonometric Functions Practice

In each of the following triangles, solve for the unknown side and angles.

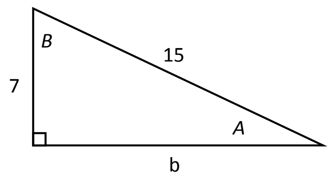
1.



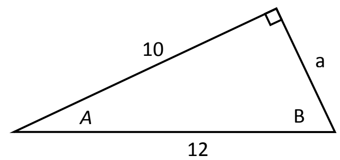
2.



3.



4.



Find a possible formula for the trigonometric function whose values are in the following tables.

5.

x	0	1	2	3	4	5	6
y	-2	4	10	4	-2	4	10

6.

x	0	1	2	3	4	5	6
y	1	-3	-7	-3	1	-3	-7

7. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature for the day is 63 degrees and the low temperature of 37 degrees occurs at 5 AM. Assuming t is the number of hours since midnight, find an equation for the temperature, D, in terms of t.

8. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature for the day is 92 degrees and the low temperature of 78 degrees occurs at 4 AM. Assuming t is the number of hours since midnight, find an equation for the temperature, D, in terms of t.

9. A population of rabbits oscillates 25 above and below an average of 129 during the year, hitting the lowest value in January ($t = 0$).

a. Find an equation for the population, P, in terms of the months since January, t.
b. What if the lowest value of the rabbit population occurred in April instead?

10. A population of elk oscillates 150 above and below an average of 720 during the year, hitting the lowest value in January ($t = 0$).

a. Find an equation for the population, P, in terms of the months since January, t.
b. What if the lowest value of the elk population occurred in March instead?

11. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature of 105 degrees occurs at 5 PM and the average temperature for the day is 85 degrees. Find the temperature, to the nearest degree, at 9 AM.

12. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the high temperature of 84 degrees occurs at 6 PM and the average temperature for the day is 70 degrees. Find the temperature, to the nearest degree, at 7 AM.

13. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature varies between 47 and 63 degrees during the day and the average daily temperature first occurs at 10 AM. How many hours after midnight does the temperature first reach 51 degrees?

14. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature varies between 64 and 86 degrees during the day and the average daily temperature first occurs at 12 AM. How many hours after midnight does the temperature first reach 70 degrees?

15. A Ferris wheel is 20 meters in diameter and boarded from a platform that is 2 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 6 minutes. How many minutes of the ride are spent higher than 13 meters above the ground?

16. A Ferris wheel is 45 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. How many minutes of the ride are spent higher than 27 meters above the ground?

17. The sea ice area around the North Pole fluctuates between about 6 million square kilometers in September to 14 million square kilometers in March. Assuming sinusoidal fluctuation, during how many months are there less than 9 million square kilometers of sea ice?

18. The sea ice area around the South Pole fluctuates between about 18 million square kilometers in September to 3 million square kilometers in March. Assuming sinusoidal fluctuation, during how many months are there more than 15 million square kilometers of sea ice?

19. A respiratory ailment called "Cheyne-Stokes Respiration" causes the volume per breath to increase and decrease in a sinusoidal manner, as a function of time. For one particular patient with this condition, a machine begins recording a plot of volume per breath versus time (in seconds). Let $b(t)$ be a function of time t that tells us the volume (in liters) of a breath that starts at time t. During the test, the smallest volume per breath is 0.6 liters and this first occurs for a breath that starts 5 seconds into the test. The largest volume per breath is 1.8 liters and this first occurs for a breath beginning 55 seconds into the test.

a. Find a formula for the function $b(t)$ whose graph will model the test data for this patient.

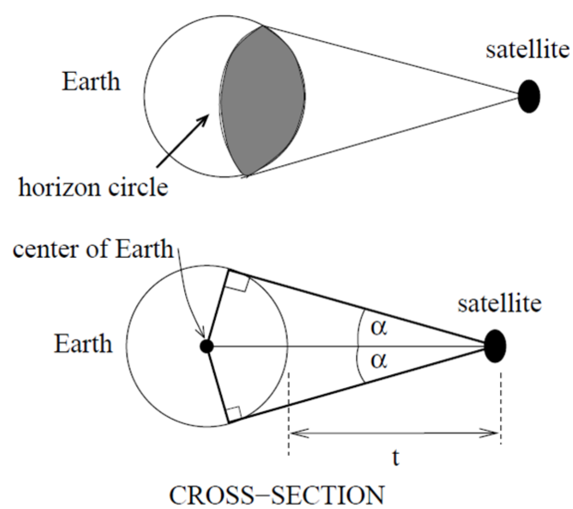
b. If the patient begins a breath every 5 seconds, what are the breath volumes during the first minute of the test?

20. Suppose the high tide in Seattle occurs at 1:00 a.m. and 1:00 p.m, at which time the water is 10 feet above the height of low tide. Low tides occur 6 hours after high tides. Suppose there are two high tides and two low tides every day and the height of the tide varies sinusoidally.

a. Find a formula for the function $y = h(t)$ that computes the height of the tide above low tide at time t. (In other words, $y = 0$ corresponds to low tide.)

b. What is the tide height at 11:00 a.m.?

21. A communications satellite orbits the earth t miles above the surface. Assume the radius of the earth is 3,960 miles. The satellite can only "see" a portion of the earth's surface, bounded by what is called a horizon circle. This leads to a two-dimensional cross-sectional picture we can use to study the size of the horizon slice:



a. Find a formula for α in terms of t.

b. If $t = 30{,}000$ miles, what is α? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

c. If $t = 1{,}000$ miles, what is α? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

d. Suppose you wish to place a satellite into orbit so that 20% of the circumference is covered by the satellite. What is the required distance t?

22. Tiffany is a model rocket enthusiast. She has been working on a pressurized rocket filled with nitrous oxide. According to her design, if the atmospheric pressure exerted on the rocket is less than 10 pounds/sq.in., the nitrous oxide chamber inside the rocket will explode. Tiff worked from a formula $p = 14.7e^{-h/10}$ pounds/sq.in. for the atmospheric pressure h miles above sea level. Assume that the rocket is launched at an angle of α above level ground at sea level with an initial speed of 1400 feet/sec. Also, assume the height (in feet) of the rocket at time t seconds is given by the equation $y(t) = -16t^2 + 1600\sin(\alpha)t$.

a. At what altitude will the rocket explode?

b. If the angle of launch is $\alpha = 12°$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?

c. If the angle of launch is $\alpha = 82°$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?

d. Find the largest launch angle α so that the rocket will not explode.

6.3 Solving Trigonometric Equations with Identities

In section 6.1, we solved basic trigonometric equations. In this section, we explore the techniques needed to solve more complicated trig equations.

Building from what we already know makes this a much easier task.

Consider the function $f(x) = 2x^2 + x$. If you were asked to solve $f(x) = 0$, it require simple algebra:

$2x^2 + x = 0$	Factor
$x(2x+1) = 0$	Set each factor equal to zero
$x = 0 \quad \text{or} \quad 2x + 1 = 0$	Solve for x
$x = 0, -\frac{1}{2}$	Final answer

Similarly, for $g(t) = \sin(t)$, if we asked you to solve $g(t) = 0$, you can solve this using unit circle values:

$$\sin(t) = 0 \quad \text{for} \quad t = 0, \pi, 2\pi, \text{and so on}$$

Using these same concepts, we consider the composition of these two functions:

$$f\big(g(t)\big) = 2(\sin(t))^2 + (\sin(t)) = 2\sin^2(t) + \sin(t)$$

This creates an equation that is a polynomial trig function. With these types of functions, we use algebraic techniques like factoring and the quadratic formula, along with trigonometric identities and techniques, to solve equations.

As a reminder, here are some of the essential trigonometric identities that we have learned so far:

Identities

Pythagorean Identities

$$\sin^2(t)+\cos^2(t)=1 \qquad 1+\cot^2(t)=\csc^2(t) \qquad \tan^2(t)+1=\sec^2(t)$$

Negative Angle Identities

$$\sin(-t)=-\sin(t) \qquad \cos(-t)=\cos(t) \qquad \tan(-t)=-\tan(t)$$

$$\csc(-t)=-\csc(t) \qquad \sec(-t)=\sec(t) \qquad \cot(-t)=-\cot(t)$$

Reciprocal Identities

$$\sec(t)=\frac{1}{\cos(t)} \qquad \csc(t)=\frac{1}{\sin(t)} \qquad \tan(t)=\frac{\sin(t)}{\cos(t)} \qquad \cot(t)=\frac{1}{\tan(t)}$$

Example 1: Solve $2\sin^2(t)+\sin(t)=0$ for all solutions with $0\le t<2\pi$.

This equation kind of looks like a quadratic equation, but with $\sin(t)$ in place of an algebraic variable (we often call such an equation "quadratic in sine"). As with all quadratic equations, we can use factoring techniques or the quadratic formula. This expression factors nicely, so we proceed by factoring out the common factor of $\sin(t)$:

$$2\sin^2(t)+\sin(t)=0 \qquad \text{Factor } \sin(t)$$

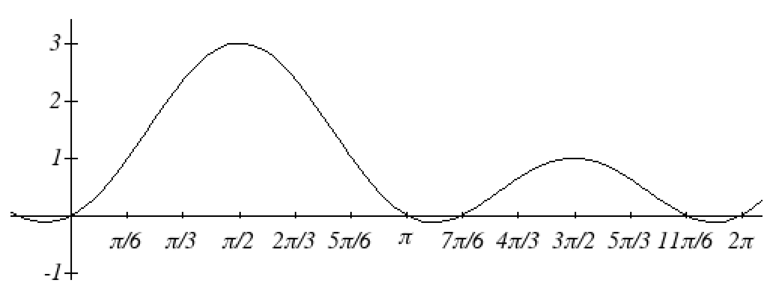
$$\sin(t)\,[2\sin(t)+1]=0 \qquad \text{Set each factor equal to zero}$$

$$\sin(t)=0 \quad \text{or} \quad 2\sin(t)+1=0 \qquad \text{Solve each for } \sin(t)$$

$$\sin(t)=0 \quad \text{or} \quad \sin(t)=-\frac{1}{2} \qquad \text{From our special angles}$$

$$t=0,\pi,\frac{7\pi}{6},\frac{11\pi}{6} \qquad \text{Final answer}$$

We could check these answers are reasonable by graphing the function and comparing the zeros.

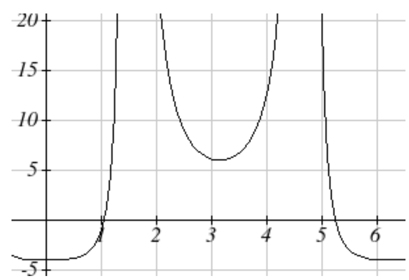


Example 2: Solve $3\sec^2(t) - 5\sec(t) - 2 = 0$ for all solutions with $0 \le t < 2\pi$.

Since the left side of this equation is quadratic in secant, we can try to factor it, and hope it factors nicely.

$3\sec^2(t) - 5\sec(t) - 2 = 0$	Factor
$(3\sec(t) + 1)(\sec(t) - 2) = 0$	Set each factor equal to zero
$3\sec(t) + 1 = 0$ or $\sec(t) - 2 = 0$	Solve for $\sec(t)$
$\sec(t) = -\frac{1}{3}$ or $\sec(t) = 2$	Rewrite as cosine
$\frac{1}{\cos(t)} = -\frac{1}{3}$ or $\frac{1}{\cos(t)} = 2$	Invert both sides
$\cos(t) = -3$ or $\cos(t) = \frac{1}{2}$	Cosine domain $[-1,1]$, only $\cos(t) = \frac{1}{2}$ has solution
$t = \frac{\pi}{3}, \frac{5\pi}{3}$	Final answer

By utilizing technology to graph the function $f(t) = 3\sec^2(t) - 5\sec(t) - 2$, a look at a graph confirms there are only two zeros for this function on the interval $[0,2\pi)$, which assures us that we didn't miss anything.



When solving some trigonometric equations, it becomes necessary to first rewrite the equation using trigonometric identities. One of the most common is the Pythagorean Identity, $\sin^2(\theta) + \cos^2(\theta) = 1$ which allows you to rewrite $\sin^2(\theta)$ in terms of $\cos^2(\theta)$ or vice versa,

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

This identity becomes very useful whenever an equation involves a combination of sine and cosine functions.

Example 3: Solve $2\sin^2(t) - \cos(t) = 1$ for all solutions with $0 \le t < 2\pi$.

Since this equation has a mix of sine and cosine functions, it becomes more complicated to solve. It is usually easier to work with an equation involving only one trig function. This is where we can use the Pythagorean Identity.

$2\sin^2(t) - \cos(t) = 1$	Use $\sin^2(\theta) = 1 - \cos^2(\theta)$
$2(1 - \cos^2(t)) - \cos(t) = 1$	Distribute
$2 - 2\cos^2(t) - \cos(t) = 1$	Subtract 1 and reorder terms
$-2\cos^2(t) - \cos(t) + 1 = 0$	Multiply by -1 to change signs
$2\cos^2(t) + \cos(t) - 1 = 0$	Factor
$(2\cos(t) - 1)(\cos(t) + 1) = 0$	Set each factor equal to zero
$2\cos(t) - 1 = 0$ or $\cos(t) + 1 = 0$	Solve for $\cos(t)$
$\cos(t) = \frac{1}{2}$ or $\cos(t) = -1$	Evaluate each special angle
$t = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$	Final answer

In addition to the Pythagorean Identity, it is often necessary to rewrite the tangent, secant, cosecant, and cotangent as part of solving an equation.

Example 4: Solve $\tan(x) = 3\sin(x)$ for all solutions with $0 \leq x < 2\pi$.

With a combination of tangent and sine, we might try rewriting tangent

$$\tan(x) = 3\sin(x) \qquad \text{Substitute } \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\frac{\sin(x)}{\cos(x)} = 3\sin(x) \qquad \text{Multiply by } \cos(x)$$

$$\sin(x) = 3\sin(x)\cos(x)$$

At this point, you may be tempted to divide both sides of the equation by $\sin(x)$. **Resist the urge**. When we divide both sides of an equation by a quantity, we are assuming the quantity is never zero. In this case, when $\sin(x) = 0$ the equation is satisfied, so we'd lose those solutions if we divided by the sine. Instead we make the equation equal zero by subtracting $3\sin(x)\cos(x)$

$$\sin(x) - 3\sin(x)\cos(x) = 0 \qquad \text{Factor GCF of } \sin(x)$$

$$\sin(x)\,(1 - 3\cos(x)) = 0 \qquad \text{Make each factor equal zero}$$

$$\sin(x) = 0 \quad \text{or} \quad 1 - 3\cos(x) = 0 \qquad \text{Solve}$$

$$\sin(x) = 0 \quad \text{or} \quad \cos(x) = \frac{1}{3} \qquad \text{The cosine we will need a calculator to evaluate}$$

$$x = \cos^{-1}\left(\frac{1}{3}\right) = 1.231 \qquad \text{Using symmetry find the second solution}$$

$$x = 2\pi - 1.231 = 5.052 \qquad \text{Evaluate } \sin(x) = 0 \text{ using special angles}$$

$$x = 0, \pi \qquad \text{List all solutions}$$

$$x = 0, \pi, 1.231, 5.052 \qquad \text{Final answer}$$

Example 5: Solve $\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$ for all solutions with $0 \le \theta < 2\pi$.

$\frac{4}{\sec^2(\theta)} + 3\cos(\theta) = 2\cot(\theta)\tan(\theta)$	Use reciprocal identities
$4\cos^2(\theta) + 3\cos(\theta) = 2 \cdot \frac{1}{\tan(\theta)}\tan(\theta)$	Simplify
$4\cos^2(\theta) + 3\cos(\theta) = 2$	Subtract 2
$4\cos^2(\theta) + 3\cos(\theta) - 2 = 0$	Can't factor, use quadratic formula
$\cos(\theta) = \frac{-3 \pm \sqrt{3^2 - 4(4)(-2)}}{2(4)} = -1.175, 0.425$	Cosine domain $[-1,1]$, only $\cos(\theta) = 0.425$ has solution
$\theta = \cos^{-1}(0.425) = 1.131$	By symmetry find second solution
$\theta = 2\pi - 1.131 = 5.152$	List answers
$\theta = 1.131, 5.152$	Final answer

6.3 Solving Trigonometric Equations with Identities Practice

Find all solutions on the interval $0 \le \theta < 2\pi$

1. $2\sin(\theta) = -1$
2. $2\sin(\theta) = \sqrt{3}$
3. $2\cos(\theta) = 1$
4. $2\cos(\theta) = -\sqrt{2}$

Find all solutions

5. $2\sin\left(\frac{\pi}{4}x\right) = 1$
6. $2\sin\left(\frac{\pi}{3}x\right) = \sqrt{2}$
7. $2\cos(2t) = -\sqrt{3}$
8. $2\cos(3t) = -1$
9. $3\cos\left(\frac{\pi}{5}x\right) = 2$
10. $8\cos\left(\frac{\pi}{2}x\right) = 6$
11. $7\sin(3t) = -2$
12. $4\sin(4t) = 1$

Find all solutions on the interval $[0,2\pi)$

13. $10\sin(x)\cos(x) = 6\cos(x)$
14. $-3\sin(t) = 15\cos(t)\sin(t)$
15. $\csc(2x) - 9 = 0$
16. $\sec(2\theta) = 3$
17. $\sec(x)\sin(x) - 2\sin(x) = 0$
18. $\tan(x)\sin(x) - \sin(x) = 0$
19. $\sec^2(x) = \frac{1}{4}$
20. $\cos^2(\theta) = \frac{1}{2}$
21. $\sec^2(x) = 7$
22. $\csc^2(t) = 3$
23. $2\sin^2(w) + 3\sin(w) + 1 = 0$
24. $8\sin^2(x) + 6\sin(x) + 1 = 0$
25. $2\cos^2(t) + \cos(t) = 1$
26. $8\cos^2(\theta) = 3 - 2\cos(\theta)$
27. $4\cos^2(x) - 4 = 15\cos(x)$
28. $9\sin(w) - 2 = 4\sin^2(w)$
29. $12\sin^2(t) + \cos(t) - 6 = 0$
30. $6\cos^2(x) + 7\sin(x) - 8 = 0$
31. $\cos^2(\phi) = -6\sin(\phi)$
32. $\sin^2(t) = \cos(t)$
33. $\tan^3(x) = 3\tan(x)$
34. $\cos^3(t) = -\cos(t)$
35. $\tan^5(x) = \tan(x)$
36. $\tan^5(x) - 9\tan(x) = 0$
37. $4\sin(x)\cos(x) + 2\sin(x) - 2\cos(x) = 1$
38. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) = 1$
39. $\tan(x) - 3\sin(x) = 0$
40. $3\cos(x) = \cot(x)$
41. $2\tan^2(t) = 3\sec(t)$
42. $1 - 2\tan(w) = \tan^2(w)$

6.4 Addition and Subtraction Identities

In this section, we begin expanding our repertoire of trigonometric identities.

Identities

The sum and difference identities

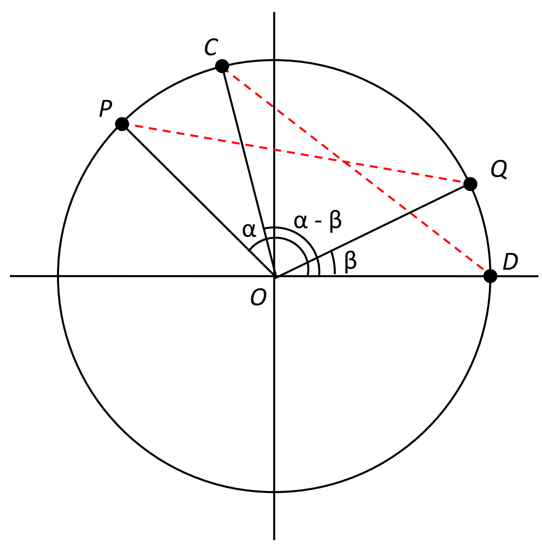
$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

We will prove the difference of angles identity for cosine. The rest of the identities can be derived from this one.



Proof of the difference of angles identity for cosine

Consider two points on a unit circle:

P at an angle of α from the positive x axis with coordinates $(\cos(\alpha), \sin(\alpha))$

Q at an angle of β with coordinates $(\cos(\beta), \sin(\beta))$

Notice the measure of angle POQ is $\alpha - \beta$. Label two more points:

C at an angle of $\alpha - \beta$, with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$,

D at the point $(1,0)$.

Notice that the distance from C to D is the same as the distance from P to Q because triangle COD is a rotation of triangle POD.

Using the distance formula to find the distance from P to Q yields

$$\sqrt{(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) - \sin(\beta))^2}$$

Expanding this

$$\sqrt{\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)}$$

Applying the Pythagorean Identity and simplifying

$$\sqrt{2 - 2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta)}$$

Similarly, using the distance formula to find the distance from C to D

$$\sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2}$$

Expanding this

$$\sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}$$

Applying the Pythagorean Identity and simplifying

$$\sqrt{-2\cos(\alpha - \beta) + 2}$$

Since the two distances are the same we set these two formulas equal to each other and simplify

$$\sqrt{2 - 2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta)} = \sqrt{-2\cos(\alpha - \beta) + 2} \qquad \text{Square both sides}$$

$$2 - 2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta) = -2\cos(\alpha - \beta) + 2 \qquad \text{Divide terms by } -2$$

$$1 + \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) + 1 \qquad \text{Subtract 1}$$

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) \qquad \text{Identity}$$

By writing $\cos(\alpha + \beta)$ as $\cos(\alpha - (-\beta))$, we can show the sum of angles identity for cosine follows from the difference of angles identity proven above.

The sum and difference of angles identities are often used to rewrite expressions in other forms, or to rewrite an angle in terms of simpler angles.

Example 1: Find the exact value of $\cos(75°)$.

$\cos(75°)$	Write 75 as $30 + 45$
$\cos(30° + 45°)$	Use sum of angles formula
$\cos(30°)\cos(45°) - \sin(30°)\cos(45°)$	Evaluate
$\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$	Simplify
$\frac{\sqrt{6} - \sqrt{2}}{4}$	Final answer

Additionally, these identities can be used to simplify expressions or prove new identities

Example 2: Prove $\frac{\tan(a)+\tan(b)}{\tan(a)-\tan(b)} = \frac{\sin(a+b)}{\sin(a-b)}$

As with any identity, we need to first decide which side to begin with. Let's start with the left side.

$\frac{\tan(a) + \tan(b)}{\tan(a) - \tan(b)}$	Rewrite with sine and cosine
$\frac{\frac{\sin(a)}{\cos(a)} + \frac{\sin(b)}{\cos(b)}}{\frac{\sin(a)}{\cos(a)} - \frac{\sin(b)}{\cos(b)}}$	Multiply top and bottom by $\cos(a)\cos(b)$
$\frac{\left(\frac{\sin(a)}{\cos(a)} + \frac{\sin(b)}{\cos(b)}\right) \cdot \cos(a)\cos(b)}{\left(\frac{\sin(a)}{\cos(a)} - \frac{\sin(b)}{\cos(b)}\right) \cdot \cos(a)\cos(b)}$	Distribute and simplify
$\frac{\sin(a)\cos(b) + \sin(b)\cos(a)}{\sin(a)\cos(b) - \sin(b)\cos(a)}$	Use sum and difference of angles
$\frac{\sin(a + b)}{\sin(a - b)}$	Final answer

These identities can also be used to solve equations.

Example 3: Solve $\sin(x)\sin(2x) + \cos(x)\cos(2x) = \frac{\sqrt{3}}{2}$.

By recognizing the left side of the equation as the result of the difference of angles identity for cosine, we can simplify the equation

$\sin(x)\sin(2x) + \cos(x)\cos(2x) = \frac{\sqrt{3}}{2}$	Apply difference of angles
$\cos(x - 2x) = \frac{\sqrt{3}}{2}$	Simplify
$\cos(-x) = \frac{\sqrt{3}}{2}$	Use negative angle identity
$\cos(x) = \frac{\sqrt{3}}{2}$	Evaluate for both values on one cycle
$x = \frac{\pi}{6}, \frac{11\pi}{6}$	For all values, where k is an integer
$x = \frac{\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k$	Final answer

Combining Waves of Equal Period

A sinusoidal function of the form $f(x) = A\sin(Bx + C)$ can be rewritten using the sum of angles identity.

Example 4: Rewrite $f(x) = 4\sin\left(3x + \frac{\pi}{3}\right)$ as a sum of sine and cosine.

$4\sin\left(3x + \frac{\pi}{3}\right)$	Sum of angles formula
$4\left[\sin(3x)\cos\left(\frac{\pi}{3}\right) + \cos(3x)\sin\left(\frac{\pi}{3}\right)\right]$	Evaluate sine and cosine
$4\left[\sin(3x)\left(\frac{1}{2}\right) + \cos(3x)\left(\frac{\sqrt{3}}{2}\right)\right]$	Distribute and simplify
$2\sin(3x) + 2\sqrt{3}\cos(3x)$	Final answer

Notice that the result is a stretch of the sine added to a different stretch of the cosine, but both have the same horizontal compression, which results in the same period.

We might ask now whether this process can be reversed – can a combination of a sine and cosine of the same period be written as a single sinusoidal function? To explore this, we will look in general at the procedure used in the example above.

$$f(x) = A\sin(Bx + C) \qquad \text{Sum of angles}$$

$$A[\sin(Bx)\cos(C) + \cos(Bx)\sin(C)] \qquad \text{Distribute}$$

$$A\sin(Bx)\cos(C) + A\cos(Bx)\sin(C) \qquad \text{Rearrange}$$

$$A\cos(C)\sin(Bx) + A\sin(C)\cos(Bx)$$

Based on this result, if we have an expression of the form $m\sin(Bx) + n\cos(Bx)$, we could rewrite it as a single sinusoidal function if we can find values A and C so that $m\sin(Bx) + n\cos(Bx) = A\cos(C)\sin(Bx) + A\sin(C)\cos(Bx)$, which will require that:

$$m = A\cos(C) \quad \text{which can be rewritten as} \quad \frac{m}{A} = \cos(C)$$

$$n = A\sin(C) \quad \text{which can be rewritten as} \quad \frac{n}{A} = \sin(C)$$

To find A, consider the expression $m^2 + n^2$

$$m^2 + n^2 = (A\cos(C))^2 + (A\sin(C))^2 \qquad \text{Evaluate exponents}$$

$$m^2 + n^2 = A^2\cos^2(C) + A^2\sin^2(C) \qquad \text{Factor } A^2$$

$$m^2 + n^2 = A^2(\cos^2(C) + \sin^2(C)) \qquad \text{Apply Pythagorean Identity}$$

$$m^2 + n^2 = A^2 \qquad \text{Solution for } A^2$$

Rewriting a Sum of Sine and Cosine as a Single Sine

To rewrite $m\sin(Bx) + n\cos(Bx)$ as $Asin(Bx + C)$

$$A^2 = m^2 + n^2, \cos(C) = \frac{m}{A}, \sin(C) = \frac{n}{A}$$

We can use either of the last two equations to solve for possible values of *C*. Since there will usually be two possible solutions, we will need to look at both to determine which quadrant *C* is in and determine which solution for *C* satisfies both equations.

Example 5: Rewrite $4\sqrt{3}\sin(2x) - 4\cos(2x)$ as a single sinusoidal function.

$4\sqrt{3}\sin(2x) - 4\cos(2x)$	Using formulas above,
$A^2 = \left(4\sqrt{3}\right)^2 + (-4)^2$	Simplify
$A^2 = 16(3) + 16 = 48 + 16 = 64$	Square root
$A = 8$	Use formulas for C
$\cos(C) = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$ $\sin(C) = -\frac{4}{8} = -\frac{1}{2}$	Solve both for C
From cosine: $C = \frac{\pi}{6}, \frac{11\pi}{6}$ From sine: $C = \frac{7\pi}{6}, \frac{11\pi}{6}$	Use the angle that works for both
$C = \frac{11\pi}{6}$	This gives the function
$8\sin\left(2x + \frac{11\pi}{6}\right)$	Final answer

Rewriting a combination of sine and cosine of equal periods as a single sinusoidal function provides an approach for solving some equations.

Example 6: Solve $3\sin(2x) + 4\cos(2x) = 1$ to find two positive solutions.

To approach this, since the sine and cosine have the same period, we can rewrite them as a single sinusoidal function.

$3\sin(2x) + 4\cos(2x) = 1$	Find A
$A^2 = 3^2 + 4^2 = 25, A = 5$	Find $\cos(C)$
$\cos(C) = \dfrac{3}{5}$	Use it to find possible C
$C = \cos^{-1}\left(\dfrac{3}{5}\right) = 0.927$ $2\pi - 0.927 = 5.356$	Find $\sin(C)$
$\sin(C) = \dfrac{4}{5}$	Use it to find possible C
$C = \sin^{-1}\left(\dfrac{4}{5}\right) = 0.927$ $\pi - 0.927 = 2.214$	Use the angle that works for both
$C = 0.927$	Rewrite equation
$5\sin(2x + 0.927) = 1$	Divide by 5
$\sin(2x + 0.927) = \dfrac{1}{5}$	Use substitution, $u = 2x + 0.927$
$\sin(u) = \dfrac{1}{5}$	Evaluate
$u = \sin^{-1}\left(\dfrac{1}{5}\right) = 0.201$ $u = \pi - 0.201 = 2.940$	Find a third solution
$u = 0.201 + 2\pi = 6.485$	Undo substitution and solve for each
$2x + 0.927 = 0.201$ $2x = -0.726$ $x = -0.363$	Use first two positive solutions
$2x + 0.927 = 2.940$ $2x = 2.013$ $x = 1.007$	

$$2x + 0.927 = 6.485$$
$$2x = 5.558$$
$$x = 2.779$$

$x = 1.007, 2.779$ Final answer

The Product-to-Sum and Sum-to-Product Identities

The Product-to-Sum Identities

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

We will prove the first of these, using the sum and difference of angles identities from the beginning of the section. The proofs of the other two identities are similar and are left as an exercise.

Proof of the product-to-sum identity for $sin(\alpha)cos(\beta)$

Recall the sum and difference of angles identities from earlier

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

Adding these two equations, we obtain

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$$

Dividing by 2, we establish the identity

$$\frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta)) = \sin(\alpha)\cos(\beta)$$

Example 7: Write $\sin(2t)\sin(4t)$ as a sum or difference.

$\sin(2t)\sin(4t)$	Use product-to-sum identity
$\frac{1}{2}(\cos(2t-4t)-\cos(2t+4t))$	Simplify
$\frac{1}{2}(\cos(-2t)-\cos(6t))$	Apply negative angle identity
$\frac{1}{2}(\cos(2t)-\cos(6t))$	Distribute
$\frac{1}{2}\cos(2t)-\frac{1}{2}\cos(6t)$	Final answer

The Sum-to-Product Identities

$$\sin(u)+\sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin(u)- \sin(v) = 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)$$

$$\cos(u)+\cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos(u)-\cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

We will again prove one of these and leave the rest as an exercise.

Proof of the sum-to-product identity for sine functions

We begin with the product-to-sum identity

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+b)+\sin(\alpha-\beta))$$

We define two new variables:

$$u = \alpha+\beta$$

$$v = \alpha-\beta$$

Adding these equations yields $u+v=2\alpha$, giving $\alpha = \frac{u+v}{2}$

Subtracting the equations yields $u - v = 2\beta$, or $\beta = \frac{u-v}{2}$

Substituting these expressions into the product-to-sum identity above,

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}(\sin(u) + \sin(v)) \quad \text{Multiply by 2}$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin(u) + \sin(v) \quad \text{Establishing the identity}$$

Example 8: Evaluate $\cos(15°) - \cos(75°)$.

$\cos(15°) - \cos(75°)$	Sum-to-product identity
$-2\sin\left(\frac{15° + 75°}{2}\right)\sin\left(\frac{15° - 75°}{2}\right)$	Simplify
$-2\sin(45°)\sin(-30°)$	Evaluate
$-2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)$	Simplify
$\frac{\sqrt{2}}{2}$	Final answer

Example 9: Prove the identity $\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} = -\tan(t)$

Since the left side seems more complicated, we can start there and simplify.

$\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)}$	Sum-to-product identity
$\frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)}$	Simplify
$-\frac{2\sin(3t)\sin(t)}{2\sin(3t)\cos(t)}$	Reduce
$-\frac{\sin(t)}{\cos(t)}$	Rewrite as tangent
$-\tan(t)$	Final answer

Example 10: Solve $\sin(\pi t)+\sin(3\pi t)=\cos(\pi t)$ for all solutions with $0 \le t < 2$.

In an equation like this, it is not immediately obvious how to proceed. One option would be to combine the two sine functions on the left side of the equation. Another would be to move the cosine to the left side of the equation, and combine it with one of the sines. For no particularly good reason, we'll begin by combining the sines on the left side of the equation and see how things work out.

$\sin(\pi t)+\sin(3\pi t)=\cos(\pi t)$	Apply sum to product identity
$2\sin\left(\frac{\pi t+3\pi t}{2}\right)\cos\left(\frac{\pi t-3\pi t}{2}\right)=\cos(\pi t)$	Simplify
$2\sin(2\pi t)\cos(-\pi t)=\cos(\pi t)$	Negative angle identity
$2\sin(2\pi t)\cos(\pi t)=\cos(\pi t)$	Subtract $\cos(\pi t)$
$2\sin(2\pi t)\cos(\pi t)-\cos(\pi t)=0$	Factor GCF
$\cos(\pi t)\,[2\sin(2\pi t)-1]=0$	Set each factor equal to zero
$\cos(\pi t)=0 \quad \text{or} \quad 2\sin(2\pi t)-1=0$	Solve
$\cos(\pi t)=0 \quad \text{or} \quad \sin(2\pi t)=\frac{1}{2}$	Solve cosine first, with $u=\pi t$

$$\cos(u)=0$$
$$u=\frac{\pi}{2},\frac{3\pi}{2}$$
$$\pi t=\frac{\pi}{2},\frac{3\pi}{2}$$
$$t=\frac{1}{2},\frac{3}{2}$$

Period of $P=\frac{2\pi}{\pi}=2$ this represents one cycle under $0 \le t < 2$

Solve sine next, with $u=2\pi t$

$$\sin(u)=\frac{1}{2}$$
$$u=\frac{\pi}{6},\frac{11\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$$
$$2\pi t=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$$
$$t=\frac{1}{12},\frac{5}{12},\frac{13}{12},\frac{17}{12}$$

Period of $P=\frac{2\pi}{2\pi}=1$. This represents two cycles under $0 \le t < 2$

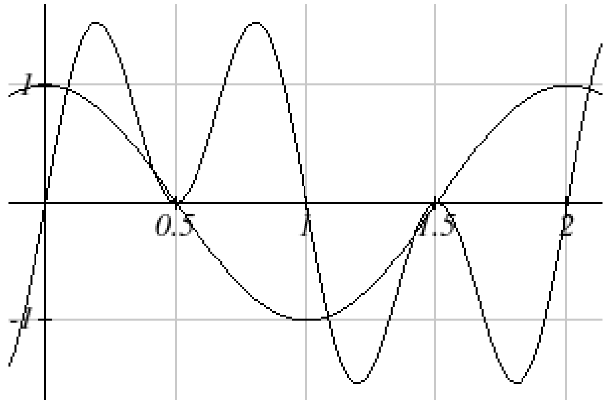
List all solutions

$$t=\frac{1}{12},\frac{5}{12},\frac{1}{2},\frac{13}{12},\frac{3}{2},\frac{17}{12}$$

Final answer

Altogether, we found six solutions on $0 \le t < 2$, which we can confirm by looking at the graph.

$$t = \frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{13}{12}, \frac{3}{2}, \frac{17}{12}$$



6.4 Addition and Subtraction Identities Practices

Find an exact value for each of the following

1. $\sin(75°)$
2. $\sin(195°)$
3 $\cos(165°)$
4. $\cos(345°)$
5. $\cos\left(\frac{7\pi}{12}\right)$
6. $\cos\left(\frac{\pi}{12}\right)$
7. $\sin\left(\frac{5\pi}{12}\right)$
8. $\sin\left(\frac{11\pi}{12}\right)$

Rewrite in terms of $\sin(x)$ and $\cos(x)$

9. $\sin\left(x+\frac{11\pi}{6}\right)$
10. $\sin\left(x-\frac{3\pi}{4}\right)$
11. $\cos\left(x-\frac{5\pi}{6}\right)$
12. $\cos\left(x+\frac{2\pi}{3}\right)$

Simplify each expression

13. $\csc\left(\frac{\pi}{2}-t\right)$
14. $\sec\left(\frac{\pi}{2}-w\right)$
15. $\cot\left(\frac{\pi}{2}-x\right)$
16. $\tan\left(\frac{\pi}{2}-x\right)$

Rewrite the product as a sum

17. $16\sin(16x)\sin(11x)$
18. $20\cos(36t)\cos(6t)$
19. $2\sin(5x)\cos(3x)$
20. $10\cos(5x)\sin(10x)$

Rewrite the sum as a product

21. $\cos(6t)+\cos(4t)$
22. $\cos(5u)+\cos(3u)$
23. $\sin(3x)+\sin(7x)$
24. $\sin(h)-\sin(3h)$

25. Given $\sin(a)=\frac{2}{3}$ and $\cos(b)=-\frac{1}{4}$ with a and b both in the interval $\left[\frac{\pi}{2},\pi\right)$ find:

a. $\sin(a+b)$

b. $\cos(a-b)$

26. Given $\sin(a)=\frac{4}{5}$ and $\cos(b)=1/3$ with a and b both in the interval $\left[0,\frac{\pi}{2}\right)$ find:

a. $\sin(a-b)$

b. $\cos(a+b)$

Solve each equation for all solutions

27. $\sin(3x)\cos(6x)-\cos(3x)\sin(6x)=-0.9$

28. $\sin(6x)\cos(11x)-\cos(6x)\sin(11x)=-0.1$

29. $\cos(2x)\cos(x) + \sin(2x)\sin(x) = 1$

30. $\cos(5x)\cos(3x) - \sin(5x)\sin(3x) = \dfrac{\sqrt{3}}{2}$

31. $\cos(5x) = -\cos(2x)$

32. $\sin(5x) = \sin(3x)$

33. $\cos(6\theta) - \cos(2\theta) = \sin(4\theta)$

34. $\cos(8\theta) - \cos(2\theta) = \sin(5\theta)$

Rewrite as a single function of the form $A\sin(Bx + C)$

35. $4\sin(x) - 6\cos(x)$

36. $-\sin(x) - 5\cos(x)$

37. $5\sin(3x) + 2\cos(3x)$

38. $-3\sin(5x) + 4\cos(5x)$

Solve for the first two positive solutions

39. $-5\sin(x) + 3\cos(x) = 1$

40. $3\sin(x) + \cos(x) = 2$

41. $3\sin(2x) - 5\cos(2x) = 3$

42. $-3\sin(4x) - 2\cos(4x) = 1$

Simplify

43. $\dfrac{\sin(7t) + \sin(5t)}{\cos(7t) + \cos(5t)}$

44. $\dfrac{\sin(9t) - \sin(3t)}{\cos(9t) + \cos(3t)}$

Prove the identity

45. $\tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{\tan(x) + 1}{1 - \tan(x)}$

46. $\tan\left(\dfrac{\pi}{4} - t\right) = \dfrac{1 - \tan(t)}{1 + \tan(t)}$

47. $\cos(a + b) + \cos(a - b) = 2\cos(a)\cos(b)$

48. $\dfrac{\cos(a + b)}{\cos(a - b)} = \dfrac{1 - \tan(a)\tan(b)}{1 + \tan(a)\tan(b)}$

49. $\dfrac{\tan(a + b)}{\tan(a - b)} = \dfrac{\sin(a)\cos(a) + \sin(b)\cos(b)}{\sin(a)\cos(a) - \sin(b)\cos(b)}$

50. $2\sin(a+b)\sin(a-b) = \cos(2b) - \cos(2a)$

51. $\dfrac{\sin(x)+\sin(y)}{\cos(x)+\cos(y)} = \tan\left(\dfrac{1}{2}(x+y)\right)$

52. $\dfrac{\cos(a+b)}{\cos(a)\cos(b)} = 1 - \tan(a)\tan(b)$

53. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

6.5 Double Angle Identities

Two special cases of the sum of angles identities arise often enough that we choose to state these identities separately.

The double angle identities

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\begin{aligned} cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\ &= 1 - 2\sin^2(\alpha) \\ &= 2\cos^2(\alpha) - 1 \end{aligned}$$

These identities follow from the sum of angles identities.

Proof of the sine double angle identity

$\sin(2\alpha)$	Rewrite
$\sin(\alpha + \alpha)$	Sum of angles identity
$\sin(\alpha)\cos(\alpha) + \cos(\alpha)\sin(\alpha)$	
$2\sin(\alpha)\cos(\alpha)$	Establishing the identity

Similarly we can show $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ by using the sum of angles identity for cosine.

For the cosine double angle identity, there are three forms of the identity stated because the basic form, $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$, can be rewritten using the Pythagorean Identity. Rearranging the Pythagorean Identity results in the equality $\cos^2(\alpha) = 1 - \sin^2(\alpha)$, and by substituting this into the basic double angle identity, we obtain the second form of the double angle identity.

$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$	Substitute the Pythagorean identity
$\cos(2\alpha) = 1 - \sin^2(\alpha) - \sin^2(\alpha)$	Simplify
$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$	Establishing the identity

Example 1: If $\sin(\theta) = \frac{3}{5}$ and θ is in the second quadrant, find $\sin(2\theta)$ and $\cos(2\theta)$.

To evaluate $\cos(2\theta)$, since we know the value for $\sin(\theta)$, we can use the version of the double angle that only involves sine. In the second quadrant this value should be positive.

$\cos(2\theta) = 1 - 2\sin^2(\theta)$	Substitute value for $\sin(\theta)$
$1 - 2\left(\frac{3}{5}\right)^2$	Evaluate exponent
$1 - 2\left(\frac{9}{25}\right)$	Multiply
$1 - \frac{18}{25}$	Simplify
$\frac{7}{25}$	Final answer

Since the double angle for sine involves both sine and cosine, we'll need to first find $\cos(\theta)$, which we can do using the Pythagorean Identity.

$\sin^2(\theta) + \cos^2(\theta) = 1$	Substitute value for $\sin(\theta)$
$\left(\frac{3}{5}\right)^2 + \cos^2(\theta) = 1$	Evaluate exponent
$\frac{9}{25} + \cos^2(\theta) = 1$	Subtract $\frac{9}{25}$
$\cos^2(\theta) = \frac{16}{25}$	Square root
$\cos(\theta) = \pm\frac{4}{5}$	In the second quadrant, cosine is negative
$\cos(\theta) = -\frac{4}{5}$	Evaluate $\sin(2\theta)$
$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$	Substitute values
$2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$	Final answer

Example 2: Simplify the expression $2\cos^2(12°) - 1$

$2\cos^2(12°) - 1$ Same form as double angle for cosine

$\cos(2 \cdot 12°) = \cos(24°)$ Final answer

Example 3: Simplify the expression $8\sin(3x)\cos(3x)$

$8\sin(3x)\cos(3x)$ Similar to double angle for sine, need factor of 2

$4 \cdot 2\sin(3x)\cos(3x)$ Use double angle identity

$4\sin(2 \cdot 3x) = 4\sin(6x)$ Final answer

We can use the double angle identities to simplify expressions and prove identities.

Example 4: Simplify $\frac{\cos(2t)}{\cos(t)-\sin(t)}$.

With three choices for how to rewrite the double angle, we need to consider which will be the most useful. To simplify this expression, it would be great if the denominator would divide out with something in the numerator, which would require a factor of $\cos(t) - \sin(t)$ in the numerator, which is most likely to occur if we rewrite the numerator with a mix of sine and cosine.

$\frac{\cos(2t)}{\cos(t) - \sin(t)}$ Apply double angle identity

$\frac{\cos^2(t) - \sin^2(t)}{\cos(t) - \sin(t)}$ Factor the numerator

$\frac{(\cos(t) - \sin(t))(\cos(t) + \sin(t))}{\cos(t) - \sin(t)}$ Divide out common factor

$\cos(t) + \sin(t)$ Final answer

Example 5: Prove $\frac{\sec^2(\alpha)}{2-\sec^2(\alpha)} = \sec(2\alpha)$.

Since the left side seems a bit more complicated than the right side, we begin there.

$$\frac{\sec^2(\alpha)}{2-\sec^2(\alpha)}$$ Rewrite as cosines

$$\frac{\frac{1}{\cos^2(\alpha)}}{2-\frac{1}{\cos^2(\alpha)}}$$ Multiply top and bottom by $\cos^2(\alpha)$

$$\frac{\frac{1}{\cos^2(\alpha)} \cdot \cos^2(\alpha)}{\left(2-\frac{1}{\cos^2(\alpha)}\right) \cdot \cos^2(\alpha)}$$ Distribute and simplify

$$\frac{1}{2\cos^2(\alpha)-1}$$ Rewrite denominator as double angle

$$\frac{1}{\cos(2\alpha)}$$ Rewrite as secant

$$\sec(2\alpha)$$ Final answer

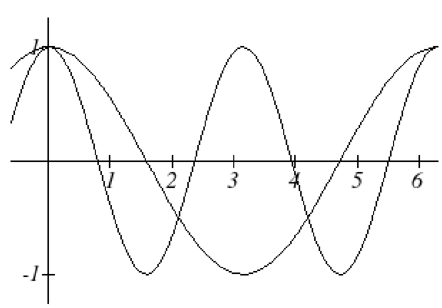
As with other identities, we can also use the double angle identities for solving equations.

Example 6: Solve $\cos(2t) = \cos(t)$ for all solutions with $0 \leq t < 2\pi$.

In general when solving trig equations, it makes things more complicated when we have a mix of sines and cosines and when we have a mix of functions with different periods. In this case, we can use a double angle identity to rewrite the $\cos(2t)$. When choosing which form of the double angle identity to use, we notice that we have a cosine on the right side of the equation. We try to limit our equation to one trig function, which we can do by choosing the version of the double angle formula for cosine that only involves cosine.

$\cos(2t) = \cos(t)$	Apply the double angle identity
$2\cos^2(t) - 1 = \cos(t)$	Subtract $\cos(t)$
$2\cos^2(t) - \cos(t) - 1 = 0$	Factor
$(2\cos(t) + 1)(\cos(t) - 1) = 0$	Set each factor equal to zero
$2\cos(t) + 1 = 0 \quad \text{or} \quad \cos(t) - 1 = 0$	Solve each
$\cos(t) = -\frac{1}{2} \quad \text{or} \quad \cos(t) = 1$	Evaluate
$t = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$	Final answer

Looking at a graph of $\cos(2t)$ and $\cos(t)$ shown together, we can verify that these three solutions on $[0,2\pi)$ seem reasonable.



Example 7: A cannonball is fired with velocity of 100 meters per second. If it is launched at an angle of θ, the vertical component of the velocity will be $100\sin(\theta)$ and the horizontal component will be $100\cos(\theta)$. Ignoring wind resistance, the height of the cannonball will follow the equation $h(t) = -4.9t^2 + 100\sin(\theta)t$ and horizontal position will follow the equation $x(t) = 100\cos(\theta)t$. If you want to hit a target 900 meters away, at what angle should you aim the cannon?

To hit the target 900 meters away, we want $x(t) = 900$ at the time when the cannonball hits the ground, when $h(t) = 0$. To solve this problem, we will first solve for the time, t, when the cannonball hits the ground. Our answer will depend upon the angle θ.

$$-4.9t^2 + 100\sin(\theta)\,t = 0$$ Factor

$$t(-4.9t + 100\sin(\theta)) = 0$$ Set each factor equal to zero

$$t = 0 \quad \text{or} \quad -4.9t + 100\sin(\theta) = 0$$ Solve

$$t = 0 \quad \text{or} \quad t = \frac{100\sin(\theta)}{4.9}$$

This shows that the height is 0 twice, once at $t = 0$ when the cannonball is fired, and again when the cannonball hits the ground after flying through the air. This second value of t gives the time when the ball hits the ground in terms of the angle θ. We want the horizontal distance $x(t)$ to be 900 when the ball hits the ground, in other words when $t = \frac{100\sin(\theta)}{4.9}$.

$$100\cos(\theta)\,t = 900$$ Substitute $t = \frac{100\sin(\theta)}{4.9}$

$$100\cos(\theta)\cdot\frac{100\sin(\theta)}{4.9} = 900$$ Simplify

$$\frac{100^2}{4.9}\sin(\theta)\cos(\theta) = 900$$ Solve for sine and cosine product

$$\sin(\theta)\cos(\theta) = \frac{900(4.9)}{100^2}$$

The left side of this equation almost looks like the result of the double angle identity for sine: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. By dividing both sides of the double angle identity by 2, we get $\frac{1}{2}\sin(2\alpha) = \sin(\alpha)\cos(\alpha)$. Applying this to the equation above,

$$\frac{1}{2}\sin(2\theta) = \frac{900(4.9)}{100^2}$$ Multiply by 2

$$\sin(2\theta) = \frac{2(900)(4.9)}{100^2}$$ Inverse sine

$$2\theta = \sin^{-1}\left(\frac{2(900)(4.9)}{100^2}\right) = 1.080$$ Divide by 2

$$\theta = 0.54 \text{ radians or } 30.94°$$ Final answer

Power Reduction and Half Angle Identities

Another use of the cosine double angle identities is to use them in reverse to rewrite a squared sine or cosine in terms of the double angle. Starting with one form of the cosine double angle identity:

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$ Add 1

$$\cos(2\alpha) + 1 = 2\cos^2(\alpha)$$ Divide by 2

$$\frac{\cos(2\alpha) + 1}{2} = \cos^2(\alpha)$$ This is called a **power reduction identity**

Similarly we can use another form of the cosine double angle identity to prove the identity $\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$.

Example 8: Rewrite $\cos^4(x)$ without any powers.

$\cos^4(x)$	Rewrite as a squared cosine squared
$(\cos^2(x))^2$	Apply power reduction identity
$\left(\dfrac{\cos(2x)+1}{2}\right)^2$	Square numerator and denominator
$\dfrac{\cos^2(2x)+2\cos(2x)+1}{4}$	Split apart the fraction
$\dfrac{\cos^2(2x)}{4}+\dfrac{\cos(2x)}{2}+\dfrac{1}{4}$	Apply power reduction identity
$\dfrac{\frac{\cos(4x)+1}{2}}{4}+\dfrac{\cos(2x)}{2}+\dfrac{1}{4}$	Simplify
$\dfrac{\cos(4x)+1}{8}+\dfrac{\cos(2x)}{2}+\dfrac{1}{4}$	Split apart the fraction
$\dfrac{\cos(4x)}{8}+\dfrac{1}{8}+\dfrac{\cos(2x)}{2}+\dfrac{1}{4}$	Combine like terms
$\dfrac{\cos(4x)}{8}+\dfrac{\cos(2x)}{2}+\dfrac{3}{8}$	Final answer

The cosine double angle identities can also be used in reverse for evaluating angles that are half of a common angle. Building from our formula $\cos^2(\alpha)=\frac{\cos(2\alpha)+1}{2}$, if we let $\theta=2\alpha$, then $\alpha=\frac{\theta}{2}$ this identity becomes $\cos^2\left(\frac{\theta}{2}\right)=\frac{\cos(\theta)+1}{2}$. Taking the square root, we obtain

$$\cos\left(\frac{\theta}{2}\right)=\pm\sqrt{\frac{\cos(\theta)+1}{2}}$$

where the sign is determined by the quadrant. This is called a **half-angle identity**.

Similarly we can prove the identity:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

.Example 9: Find an exact value for $\cos(15°)$.

Since 15 degrees is half of 30 degrees, we can use our result from above:

$$\cos(15°) = \cos\left(\frac{30°}{2}\right) = \pm\sqrt{\frac{\cos(30°)+1}{2}}$$

We can evaluate the cosine. Since 15 degrees is in the first quadrant, we need the positive result.

$$\sqrt{\frac{\cos(30°)+1}{2}} = \sqrt{\frac{\frac{\sqrt{3}}{2}+1}{2}} = \sqrt{\frac{\sqrt{3}}{4}+\frac{1}{2}}$$

Below is a summary of the identities from this section

Half-Angle Identities

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)+1}{2}} \qquad \sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

Power Reduction Identities

$$\cos^2(\alpha) = \frac{\cos(2\alpha)+1}{2} \qquad \sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}$$

Since these identities are easy to derive from the double-angle identities, the power reduction and half-angle identities are not ones you should need to memorize separately.

6.5 Double Angle Identities Practice

1. If $\sin(x) = \frac{1}{8}$ and x is in quadrant I, then find the exact values for (without solving for x)

   a. $\sin(2x)$ b. $\cos(2x)$ c. $\tan(2x)$

2. If $\cos(x) = \frac{2}{3}$ and x is in quadrant I, then find the exact values for (without solving for x)

   a. $\sin(2x)$ b. $\cos(2x)$ c. $\tan(2x)$

Simplify each expression

3. $\cos^2(28°) - \sin^2(28°)$
4. $2\cos^2(37°) - 1$
5. $1 - 2\sin^2(17°)$
6. $\cos^2(37°) - \sin^2(37°)$
7. $\cos^2(9x) - \sin^2(9x)$
8. $\cos^2(6x) - \sin^2(6x)$
9. $4\sin(8x)\cos(8x)$
10. $6\sin(5x)\cos(5x)$

Solve for all solutions on the interval $[0,2\pi)$

11. $6\sin(2t) + 9\sin(t) = 0$
12. $2\sin(2t) + 3\cos(t) = 0$
13. $9\cos(2\theta) = 9\cos^2(\theta) - 4$
14. $8\cos(2\alpha) = 8\cos^2(\alpha) - 1$
15. $\sin(2t) = \cos(t)$
16. $\cos(2t) = \sin(t)$
17. $\cos(6x) - \cos(3x) = 0$
18. $\sin(4x) - \sin(2x) = 0$

Use a double angle, half angle, or power reduction formula to rewrite without exponents

19. $\cos^2(5x)$
20. $\cos^2(6x)$
21. $\sin^4(8x)$
22. $\sin^4(3x)$
23. $\cos^2(x)\sin^4(x)$
24. $\cos^4(x)\sin^2(x)$
25. If $\csc(x) = 7$ and $90° < x < 180°$, then find the exact values for (without solving for x)

    a. $\sin\left(\frac{x}{2}\right)$ b. $\cos\left(\frac{x}{2}\right)$ c. $\tan\left(\frac{x}{2}\right)$

26. If $\sec(x) = 4$ and $90° < x < 180°$, then find the exact values for (without solving for x)

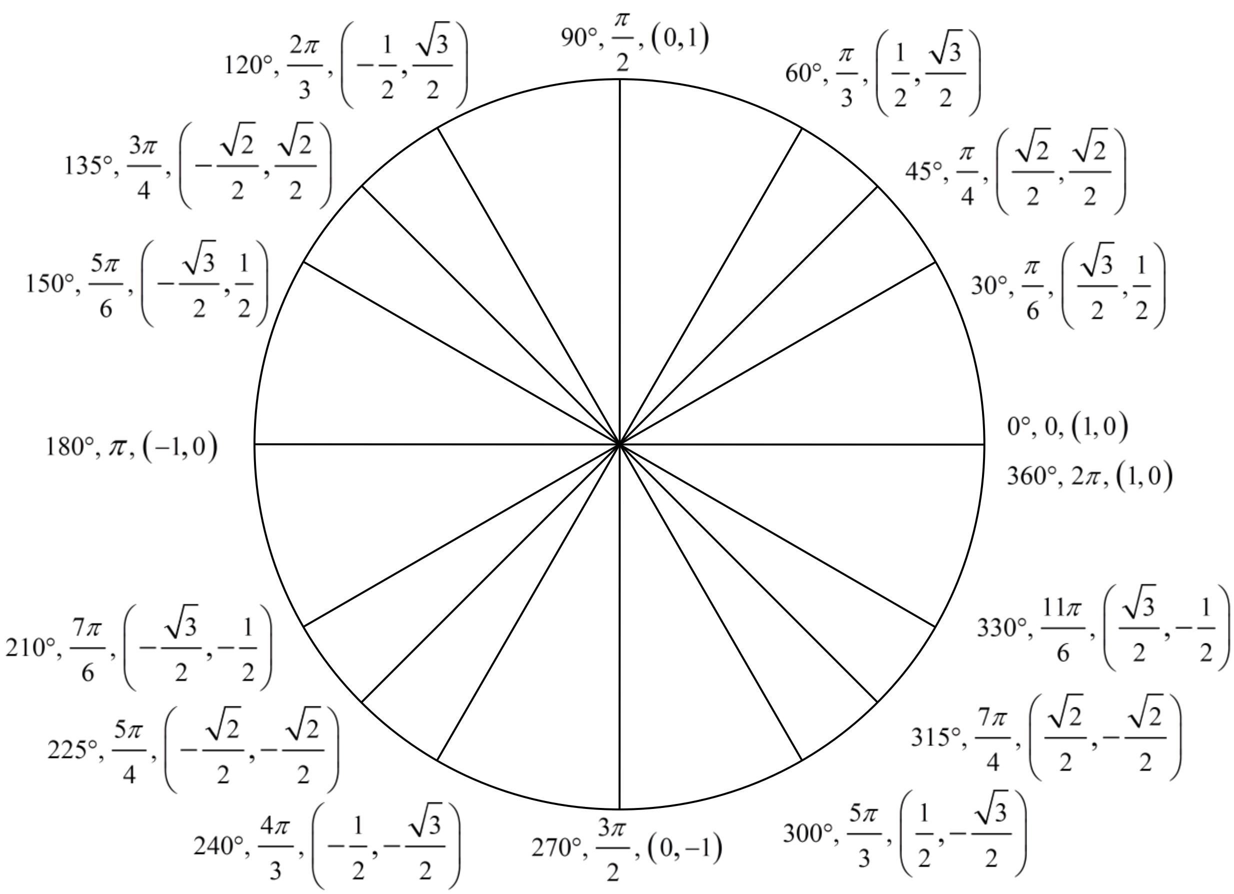
    a. $\sin\left(\frac{x}{2}\right)$ b. $\cos\left(\frac{x}{2}\right)$ c. $\tan\left(\frac{x}{2}\right)$

6.6 Review Trig Identities and Trig Equations

This section will review the use of the identities we have seen in the past two chapters.

Refer back to 6.1 and 6.3 for examples of solving trig equations.

The Unit Circle:



Cofunction Identities

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Reciprocal Identities

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Tangent/Cotangent Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Even/Odd or Negative Angle Identities

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$

$$\sec(-\theta) = \sec(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

Sum and Difference Identities

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

Product-to-Sum Identities

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

Sum-to-Product Identities

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin(u) - \sin(v) = 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)$$

$$\cos(u) + \cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Double Angle Identities

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\begin{aligned} cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\ &= 1 - 2\sin^2(\alpha) = 2\cos^2(\alpha) - 1 \end{aligned}$$

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$$

Half-Angle Identities

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta) + 1}{2}}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$

Power Reduction Identities

$$\cos^2(\alpha) = \frac{\cos(2\alpha) + 1}{2}$$

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2}$$

6.6 Review Trig Identities and Trig Equations Practice

Solve each of the following equations for $0 \le \theta < 2\pi$

1. $2\cos^2(\theta) + \cos(\theta) = 0$
2. $\sin^2(\theta) - 1 = 0$
3. $2\sin^2(\theta) - \sin(\theta) - 1 = 0$
4. $2\cos^2(\theta) + \cos(\theta) - 1 = 0$
5. $(\tan(\theta) - 1)(\sec(\theta) - 1) = 0$
6. $(\cos(\theta) + 1)\left(\csc(\theta) - \frac{1}{2}\right) = 0$
7. $\sin^2(\theta) - \cos^2(\theta) = 1 + \cos(\theta)$
8. $\cos^2(\theta) - \sin^2(\theta) + \sin(\theta) = 0$
9. $\sin^2(\theta) = 6(\cos(\theta) + 1)$
10. $2\sin^2(\theta) = 3(1 - \cos(\theta))$
11. $\cos(2\theta) + 6\sin^2(\theta) = 4$
12. $\cos(2\theta) = 2 - 2\sin^2(\theta)$
13. $\cos(\theta) = \sin(\theta)$
14. $\cos(\theta) + \sin(\theta) = 1$
15. $\tan(\theta) = 2\sin(\theta)$
16. $\sin(2\theta) = \cos(\theta)$
17. $\sin(\theta) = \csc(\theta)$
18. $\tan(\theta) = \cot(\theta)$
19. $\cos(2\theta) = \cos(\theta)$
20. $\sin(2\theta)\sin(\theta) = \cos(\theta)$
21. $\sin(2\theta) + \sin(4\theta) = 0$
22. $\cos(2\theta) + \cos(4\theta) = 0$
23. $\cos(4\theta) - \cos(6\theta) = 0$
24. $\sin(4\theta) - \sin(6\theta) = 0$
25. $1 + \sin(\theta) = 2\cos^2(\theta)$
26. $\sin^2(\theta) - 2\cos(\theta) + 2 = 0$
27. $2\sin^2(\theta) - 5\sin(\theta) + 3 = 0$
28. $2\cos^2(\theta) - 7\cos(\theta) - 4 = 0$
29. $3(1 - \cos(\theta)) = \sin^2(\theta)$
30. $4(1 + \sin(\theta)) = \cos^2(\theta)$
31. $\tan^2(\theta) = \frac{3}{2}\sec(\theta)$
32. $\csc^2(\theta) = \cot(\theta) + 1$
33. $3 - \sin(\theta) = \cos(2\theta)$
34. $\cos(2\theta) + 5\cos(\theta) + 3 = 0$
35. $\sec^2(\theta) + \tan(\theta) = 0$
36. $\sec(\theta) = \tan(\theta) + \cot(\theta)$
37. $\sin(\theta) - \sqrt{3}\cos(\theta) = 1$
38. $\sqrt{3}\sin(\theta) + \cos(\theta) = 1$
39. $\tan(2\theta) + 2\sin(\theta) = 0$
40. $\tan(2\theta) + 2\cos(\theta) = 0$
41. $\sin(\theta) + \cos(\theta) = \sqrt{2}$
42. $\sin(\theta) + \cos(\theta) = -\sqrt{2}$

Prove the following identities

43. $\csc(\theta) - \cos(\theta)\cot(\theta) = \sin(\theta)$

44. $\sec(\theta) - \sin(\theta)\tan(\theta) = \cos(\theta)$

45. $\dfrac{1+\cos(\theta)}{\sin(\theta)} + \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{\cos(\theta)+1}{\sin(\theta)\cos(\theta)}$

46. $\dfrac{1}{\sin(\theta)\cos(\theta)} - \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{\sin(\theta)\cos(\theta)}{1-\sin^2(\theta)}$

47. $\dfrac{1-\sin(\theta)}{\cos(\theta)} = \dfrac{\cos(\theta)}{1+\sin(\theta)}$

48. $\dfrac{1-\cos(\theta)}{\sin(\theta)} = \dfrac{\sin(\theta)}{1+\cos(\theta)}$

49. $\dfrac{1+\tan(\theta)}{1+\cot(\theta)} = \dfrac{\sec(\theta)}{\csc(\theta)}$

50. $\dfrac{\cot(\theta)-1}{1-\tan(\theta)} = \dfrac{\csc(\theta)}{\sec(\theta)}$

51. $\dfrac{\sin(\theta)+\cos(\theta)}{\sec(\theta)+\csc(\theta)} = \dfrac{\sin(\theta)}{\sec(\theta)}$

52. $\dfrac{\sin(\theta)+\cos(\theta)}{\sec(\theta)+\csc(\theta)} = \dfrac{\cos(\theta)}{\csc(\theta)}$

53. $\dfrac{1+\tan(\theta)}{1-\tan(\theta)} + \dfrac{1+\cot(\theta)}{1-\cot(\theta)} = 0$

54. $\dfrac{\cos^2(\theta)+\cot(\theta)}{\cos^2(\theta)-\cot(\theta)} = \dfrac{\cos^2(\theta)\tan(\theta)+1}{\cos^2(\theta)\tan(\theta)-1}$

55. $\dfrac{1+\cos(2\theta)}{\sin(2\theta)} = \cot(\theta)$

56. $\dfrac{2\tan(\theta)}{1+\tan^2(\theta)} = \sin(2\theta)$

57. $\dfrac{\sec^2(\theta)}{2-\sec^2(\theta)} = \sec(2\theta)$

58. $\dfrac{\cot^2(\theta)-1}{2\cot(\theta)} = \cot(2\theta)$

59. $\dfrac{\sin(\alpha+\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha)+\tan(\beta)$

60. $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha)\sin(\beta)} = \cot(\beta)-\tan(\alpha)$

61. $\dfrac{\tan(\theta)+\sin(\theta)}{2\tan(\theta)} = \cos^2\left(\dfrac{\theta}{2}\right)$

62. $\dfrac{\tan(\theta)-\sin(\theta)}{2\tan(\theta)} = \sin^2\left(\dfrac{\theta}{2}\right)$

63. $\cos^4(\theta) - \sin^4(\theta) = \cos(2\theta)$

64. $\dfrac{\cos^4(\theta)-\sin^4(\theta)}{1-\tan^4(\theta)} = \cos^4(\theta)$

65. $\dfrac{\tan(3\theta)-\tan(\theta)}{1+\tan(3\theta)\tan(\theta)} = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

66. $\left(\dfrac{1+\tan(\theta)}{1-\tan(\theta)}\right)^2 = \dfrac{1+\sin(2\theta)}{1-\sin(2\theta)}$

67. $\dfrac{\cos^3(\theta)-\sin^3(\theta)}{\cos(\theta)-\sin(\theta)} = \dfrac{2+\sin(2\theta)}{2}$

68. $\dfrac{\sin^3(\theta)+\cos^3(\theta)}{\sin(\theta)+\cos(\theta)} = \dfrac{2-\sin(2\theta)}{2}$

69. $\sec(2\theta)+\tan(2\theta)+1 = \dfrac{2}{1-\tan(\theta)}$

70. $\cos(2\theta)\,(1+\tan^2(\theta)) = 2 - \dfrac{1}{\cos^2(\theta)}$

71. $1 + \tan(\pi + 2\theta)\cot\left(\frac{\pi}{2} - \theta\right) = \sec(2\theta)$

72. $(\tan(\theta) + \sec(\theta))^2 = \frac{1 + \sin(\theta)}{1 - \sin(\theta)}$

73. $\tan\left(\frac{\pi}{6} - \theta\right)\tan\left(\frac{\pi}{6} + \theta\right) = \frac{2\cos(2\theta) - 1}{2\cos(2\theta) + 1}$

74. $\tan\left(\frac{1}{2}\theta\right) + \cot\left(\frac{1}{2}\theta\right) = 2\csc(\theta)$

75. $1 - \cos(2\theta)\sec^2(\theta) = \tan^2(\theta)$

76. $\frac{\sin(3\theta)}{\sin(\theta)} - \frac{\cos(3\theta)}{\cos(\theta)} = 2$

77. $\frac{\sqrt{1 + \sin(2\theta)} - \cos(\theta)}{\sqrt{1 + \sin(2\theta)} - \sin(\theta)} = \tan(\theta)$

78. $\frac{1}{\cot(\theta)} + \frac{1}{\tan(\theta)} = \frac{2}{\sin(2\theta)}$

79. $\tan^2(\theta)(1 + \cos(2\theta)) + 2\cos^2(\theta) = 2$

80. $\cos^6(\theta) + \sin^6(\theta) = 1 - \frac{3}{4}\sin^2(2\theta)$

81. $\tan^2(\theta)(1 + \cos(2\theta)) + 2\cos^2(\theta) = 2$

82. $\frac{\sin(\theta) + \sin(3\theta)}{2\sin(2\theta)} = \cos(\theta)$

83. $\frac{\cos(\theta) + \cos(3\theta)}{2\cos(2\theta)} = \cos(\theta)$

84. $\frac{\sin(4\theta) + \sin(2\theta)}{\cos(4\theta) + \cos(2\theta)} = \tan(3\theta)$

85. $\frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$

86. $\frac{\cos(\theta) - \cos(3\theta)}{\sin(\theta) + \sin(3\theta)} = -\tan(\theta)$

87. $\frac{\cos(\theta) - \cos(5\theta)}{\sin(\theta) + \sin(5\theta)} = \tan(2\theta)$

88. $\frac{\sin(4\theta) + \sin(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(6\theta)$

89. $\frac{\sin(4\theta) - \sin(8\theta)}{\cos(4\theta) - \cos(8\theta)} = -\cot(6\theta)$

90. $\frac{\sin(4\theta) + \sin(8\theta)}{\sin(4\theta) - \sin(8\theta)} = -\frac{\tan(6\theta)}{\tan(2\theta)}$

91. $\frac{\cos(4\theta) - \cos(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(2\theta)\tan(6\theta)$

92. $\frac{\sin(\alpha) + \sin(\beta)}{\cos(\alpha) + \cos(\beta)} = \tan\left(\frac{\alpha + \beta}{2}\right)$

93. $\frac{\sin(\alpha) - \sin(\beta)}{\cos(\alpha) - \cos(\beta)} = -\cot\left(\frac{\alpha + \beta}{2}\right)$

94. $1 - \cos(5\theta)\cos(3\theta) - \sin(5\theta)\sin(3\theta) = 2\sin^2(\theta)$

95. $2\sin(\theta)\cos^3(\theta) + 2\sin^3(\theta)\cos(\theta) = \sin(2\theta)$

96. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2(\alpha) - \sin^2(\beta)$

97. $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2(\alpha)-\sin^2(\beta)$

98. $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos(\alpha)\cos(\beta)$

99. $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin(\alpha)\cos(\beta)$

100. $\log(\cos(2\theta))-\log(1-\sin(2\theta))=\log(1+\tan(\theta))-\log(1-\tan(\theta))$

101. $\cos\left(\frac{\pi}{6}+\theta\right)-\cos\left(\frac{\pi}{6}-\theta\right)=\cos\left(\frac{\pi}{2}+\theta\right)$

102. $\log(1+\sin(2\theta)-\cos(2\theta))-\log(\sin(\theta)+\cos(\theta))=\log(2)-\log(\csc(\theta))$

103. $\cos(\theta)\,(2\sec(\theta)+\tan(\theta))(\sec(\theta)-2\tan(\theta))=2\cos(\theta)-3\tan(\theta)$

104. $\log(\cos(2\theta))-\log(1-\sin(2\theta))=\log(\cot(\theta)+1)-\log(\cot(\theta)-1)$

105. $\log(1+\tan(\theta))-\log(1-\tan(\theta))=\log(\sec(2\theta)+\tan(2\theta))$

106. $\log(1-\cos(\theta))-\log(\sin(\theta))=\log(\sin(\theta))-log(1+cos(\theta))$

107. $\sin(\theta)\,(\sin(\theta)+\sin(3\theta))=\cos(\theta)(\cos(\theta)-\cos(3\theta))$

108. $\sin(\theta)\,(\sin(3\theta)+\sin(5\theta))=\cos(\theta)\,(\cos(3\theta)-\cos(5\theta))$

109. $\dfrac{\sin(\alpha)+\sin(\beta)}{\sin(\alpha)-\sin(\beta)}=\tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

110. $\dfrac{\cos(\alpha)+\cos(\beta)}{\cos(\alpha)-\cos(\beta)}=-\cot\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$

111. $1+\cos(2\theta)+\cos(4\theta)+\cos(6\theta)=4\cos(\theta)\cos(2\theta)\cos(3\theta)$

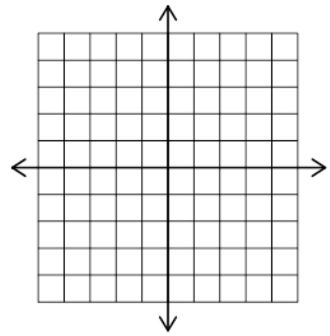
112. $1-\cos(2\theta)+\cos(4\theta)-\cos(6\theta)=4\sin(\theta)\cos(2\theta)\sin(3\theta)$

Chapter 7

Polar Coordinates

7.1 Polar Coordinates

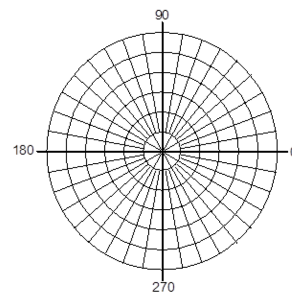
The coordinate system we are most familiar with is called the Cartesian coordinate system, a rectangular plane divided into four quadrants by the horizontal and vertical axes.



In earlier chapters, we often found the Cartesian coordinates of a point on a circle at a given angle from the positive horizontal axis. Sometimes that angle, along with the point's distance from the origin provides a more useful way of describing the point's location than the conventional Cartesian coordinates.

Polar Coordinates

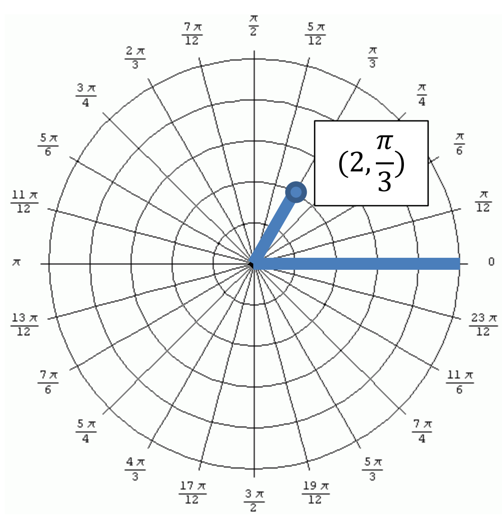
Polar coordinates of a point consist of an ordered pair (r, θ), where r is the distance from the point to the origin, and θ is the angle measured in standard position.



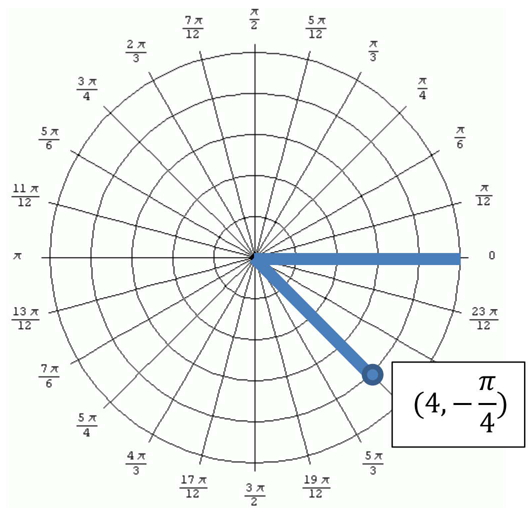
Notice that if we were to "grid" the plane from polar coordinates it would look like the graph to the right, with circles at incremental radii, and rays drawn at incremental angles.

Example 1: Graph the points whose polar coordinates are $\left(2, \frac{\pi}{3}\right)$, $\left(4, -\frac{\pi}{4}\right)$, and $\left(-5, \frac{2\pi}{3}\right)$

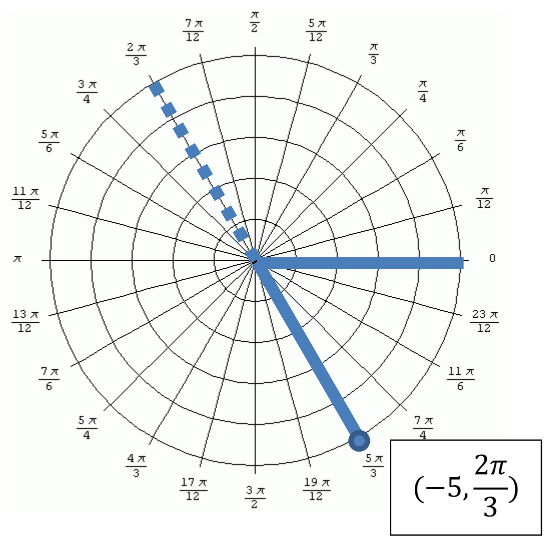
The first point is 2 units from the pole (0 radians) with a terminal side at $\theta = \frac{\pi}{3}$. (Counterclockwise)



The second point is 4 units from the pole with a terminal side at $\theta = -\frac{\pi}{4}$. (Clockwise)



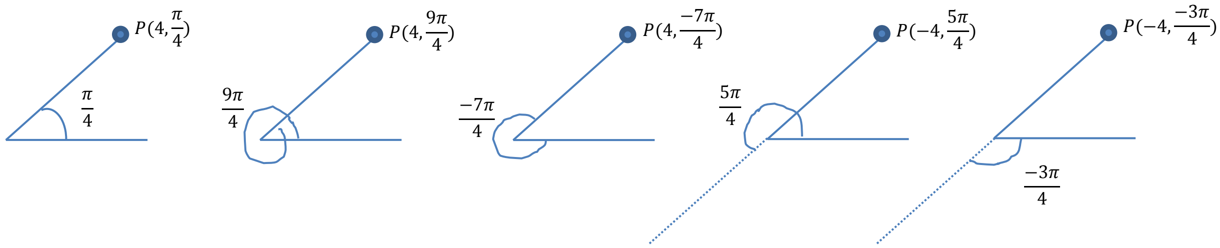
The third point has a negative radius, we will first consider an angle of $\theta = \frac{2\pi}{3}$ then have the radius of 5 be a reflection of this angle as shown below.



Another method to find the above point with a negative radius is to add π to the angle.

$$\theta + \pi = \frac{2\pi}{3} + \pi = \frac{2\pi}{3} + \frac{3\pi}{3} = \frac{5\pi}{3}$$

Then use this new angle with the positive radius. This leads to an important point about polar coordinates: The polar coordinates of a point are not unique. For example, as we see below, $\left(4, \frac{\pi}{4}\right)$, $\left(4, \frac{9\pi}{4}\right)$, $\left(4, -\frac{7\pi}{4}\right)$, $\left(-4, \frac{5\pi}{4}\right)$, and $\left(-4, -\frac{3\pi}{4}\right)$ all represent the same point p.



Converting Points

To convert between polar coordinates and Cartesian coordinates, we recall relationships we developed back in chapter 5.

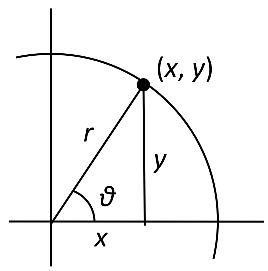
Converting Between Polar and Cartesian Coordinates

To convert between polar (r, θ) and Cartesian (x, y) coordintes, we use the relationships:

$$\cos(\theta) = \frac{x}{r} \qquad x = r\cos(\theta)$$

$$\sin(\theta) = \frac{y}{r} \qquad y = r\sin(\theta)$$

$$\tan(\theta) = \frac{y}{x} \qquad x^2 + y^2 = r^2$$



From these relationships and our knowledge of the unit circle, if $r = 1$ and $\theta = \frac{\pi}{3}$, the polar coordinates would be $(r, \theta) = (1, \frac{\pi}{3})$, and the corresponding Cartesian coordinates $(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Remembering your unit circle values will come in very handy as you convert between Cartesian and polar coordinates.

Example 2: Convert $\left(3, \frac{7\pi}{6}\right)$ from polar coordinates to rectangular coordinates.

$\left(3, \frac{7\pi}{6}\right)$	Use formulas for x and y
$x = r\cos(\theta) = 3\cos\left(\frac{7\pi}{6}\right)$ $y = r\sin(\theta) = 3\sin\left(\frac{7\pi}{6}\right)$	Evaluate each
$x = -\frac{3\sqrt{3}}{2}$ $y = -\frac{3}{2}$	Give answer as rectangular ordered pair
$\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$	Final answer

Example 3: Find the polar coordinates of the point with Cartesian coordinates $(-1, \sqrt{3})$

$(-1, \sqrt{3})$	Find r with Pythagorean theorem
$r^2 = (-1)^2 + \left(\sqrt{3}\right)^2$	Square
$r^2 = 1 + 3$	Add
$r^2 = 4$	Square root
$r = 2$	Use cosine (or sine) to find θ
$\cos(\theta) = \frac{x}{r} = -\frac{1}{2}$	Inverse cosine
$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$	The point $(-1, \sqrt{3})$ is in the second quadrant
$\theta = \frac{2\pi}{3}$	Give answer as polar ordered pair
$\left(2, \frac{2\pi}{3}\right)$	Final answer

Polar Equations

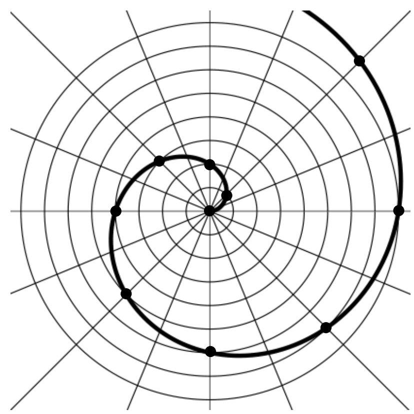
Just as a Cartesian equation like $y = x^2$ describes a relationship between x and y values on a Cartesian grid, a polar equation can be written describing a relationship between r and θ values on the polar grid.

Example 4: Sketch the graph of the polar equation $r = \theta$

The equation $r = \theta$ describes all the points for which the radius r is equal to the angle. To visualize this relationship, we can create a table of values:

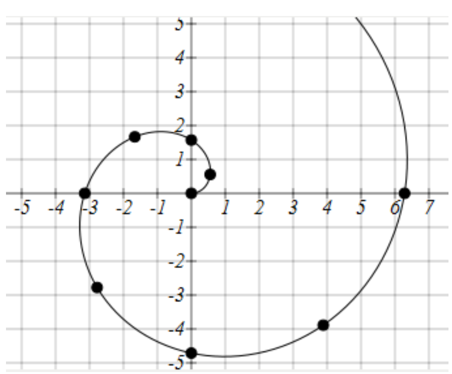
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

We can plot these points on the plane, and then sketch a curve that fits the points. The resulting graph is a spiral.



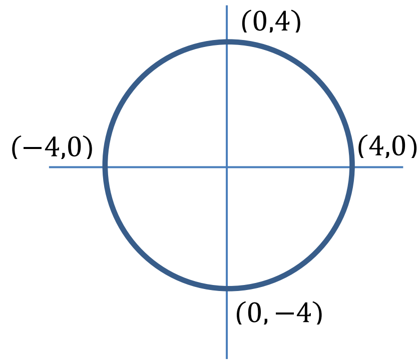
Notice that the resulting graph cannot be the result of a function of the form $y = f(x)$, as it does not pass the vertical line test, even thouth it resulted from a function giving r in terms of θ.

Although it is nice to see polar equations on polar grids, it is not uncommon for polar graphs to be graphed on the Cartesian coordinate system. For this example, the spiral graph above is also graphed on the Cartesian grid below.



Example 5: Graph the polar equation $r = 4$

Since every point satisfying the equation $r = 4$ is 4 units from the origin, we see that the graph is the circle of radius 4 centered at the origin



In general, $r = a$ is a circle centered at the origin, radius a. Also, $\theta = c$ is a line through the origin.

The normal setting on graphing calculators and software graph on the Cartesian coordinate system with y being a function of x, where the graphing utility asks for $f(x)$, or simply $y =$.

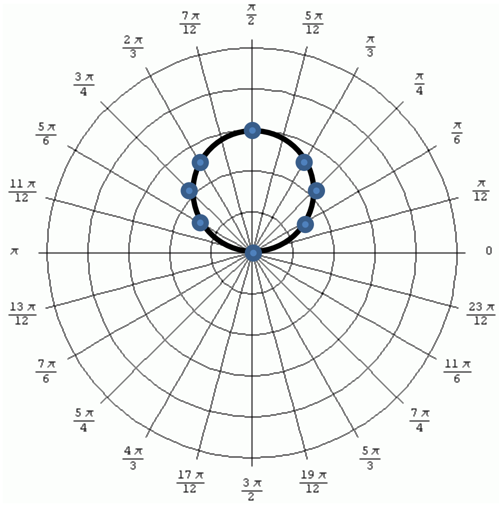
To graph polar equations, you may need to change the mode of your calculator to Polar. You will know you have been successful in changing the mode if you now have r as a function of θ, where the graphing utility asks for $r(\theta)$, or simply $r =$.

Example 6: Sketch the graph of the polar equation $r = 3\sin(\theta)$

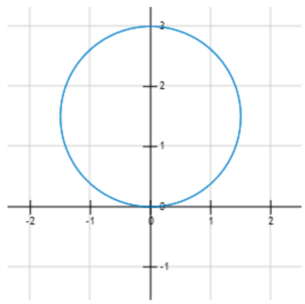
Make a table of values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$\frac{3}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{3}}{2}$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3\sqrt{2}}{2}$	$\frac{3}{2}$	0
Approx r	0	1.5	2.1	2.6	3	2.6	2.1	1.5	0

Graph each of these points on the polar grid and connect the dots:



Graphing this on the Cartesian plane gives the following graph:



In general $r = a\sin(\theta)$ and $r = a\cos(\theta)$ are circles tangent to the origin, radius $\frac{|a|}{2}$, x or y axis is the axis of symmetry

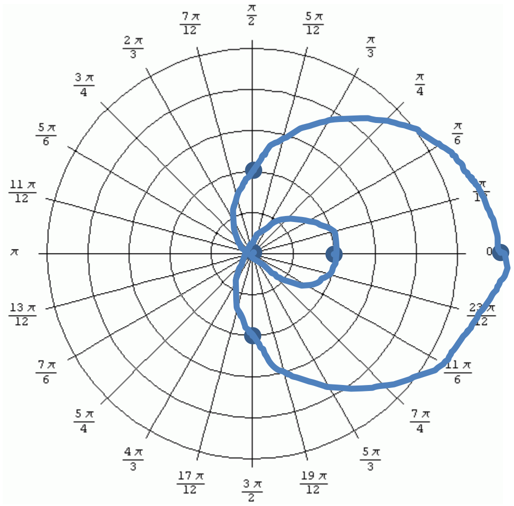
Example 7: Sketch the graph of the polar equation $r = 2 + 4\cos(\theta)$ (this graph is called a limacon)

It may be easier to first find the pole values by solving $r = 0$

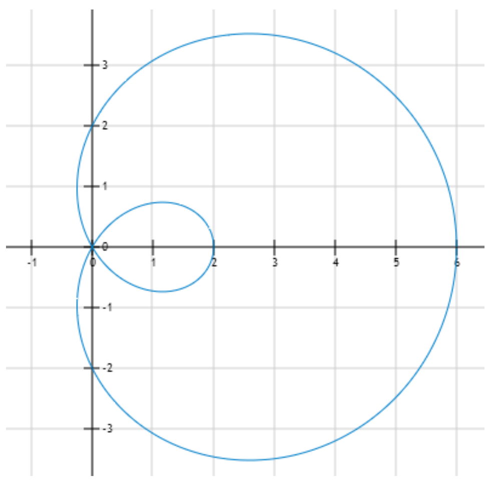
$2 + 4\cos(\theta) = 0$	Subtract 2
$4\cos(\theta) = -2$	Divide by 4
$\cos(\theta) = -\frac{1}{2}$	Inverse cosine
$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \frac{4\pi}{3}$	List the points
$\left(0, \frac{2\pi}{3}\right), \left(0, \frac{4\pi}{3}\right)$	Now make a table for other angles

θ	0	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	6	2	0	-2	0	2

Plot all points and connect in order:



Below is the same graph on rectangular coordinates:



Testing for Symmetry in Polar Coordinates

The graph of a polar equation is:

1. Symmetric with respect to the line $\theta = \frac{\pi}{2}$ if replacing (r, θ) by $(r, \pi - \theta)$ yields an equivalent equation

2. Symmetric with respect to the polar axis if replacing (r, θ) by $(r, -\theta)$ yields an equivalent equation

3. Symmetric with respect to the pole if replacing (r, θ) by $(-r, \theta)$ results in an equivalent equation

Example 8: Graph $r = 4 - 4\sin(\theta)$ (this type of graph is called cardioid)

First test for symmetry, replacing θ by $\pi - \theta$:

$r = 4 - 4\sin(\pi - \theta)$ Since $\sin(\pi - \theta) = \sin(\theta)$

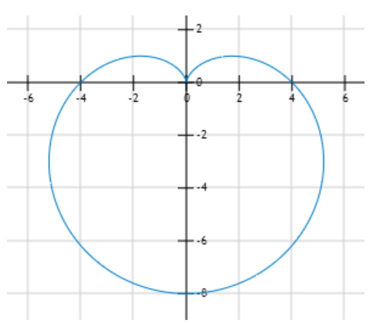
$r = 4 - 4\sin(\theta)$ Same equation!
Symmetric with respect to $\theta = \frac{\pi}{2}$ line
(y axis in rectangular)

Second test for and third test for symmetry do not obtain the same equation. Hence we can draw no conclusion about symmetry with respect to the polar axis or the pole.

As the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ we make a table of values for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	8	7.5	6.8	6	4	2	1.2	0.5	0

Graphing these points (and their reflections) yields the following graph:



In general a cardioid will have the form $r = a(1 \pm \cos(\theta))$ or $r = a(1 \pm \sin(\theta))$.

Example 9: Graph $r = 2\cos(3\theta)$ (this type of graph is called a 3 leave rose)

Checking for symmetry, because $\cos(-\theta) = \cos(\theta)$ we have symmetry with respect to the polar axis.

Next we solve for when $r = 0$

$$0 = 2\cos(3\theta) \qquad \text{Divide by 2}$$

$$0 = \cos(3\theta) \qquad \text{Inverse cosine}$$

$$3\theta = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \qquad \text{Divide by 3}$$

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \qquad \text{List coordinates}$$

$$\left(0, \frac{\pi}{6}\right), \left(0, \frac{\pi}{2}\right), \left(0, \frac{5\pi}{6}\right)$$

We also know that when $\cos(3\theta) = 1$ we end up with $r = 2$. This gives

$$\cos(3\theta) = 1 \qquad \text{Inverse cosine}$$

$$3\theta = \cos^{-1}(1) = 0, 2\pi \qquad \text{Divide by 3}$$

$$\theta = 0, \frac{2\pi}{3} \qquad \text{List coordinates}$$

$$(2,0), \left(2, \frac{2\pi}{3}\right)$$

Similarly we know when $\cos(3\theta) = -1$ we end up with $r = -2$. This gives the points:

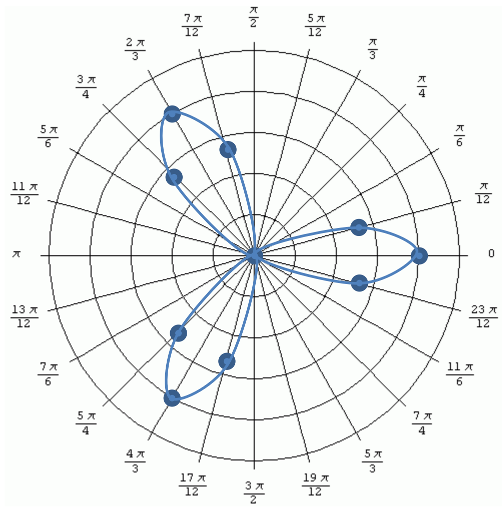
$$\left(-2, \frac{\pi}{3}\right), \left(-2, \frac{3\pi}{3}\right)$$

Making a table for a few other values to fill in the gaps gives:

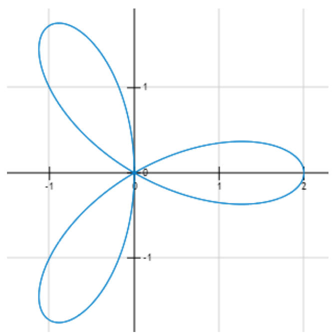
θ	$\frac{\pi}{12}$	$\frac{\pi}{4}$	$\frac{5\pi}{12}$
r	$\sqrt{2} = 1.4$	$-\sqrt{2} = -1.4$	$-\sqrt{2} = -1.4$

Graph each of the points and reflections to obtain:

Scale: Each circle is $\frac{1}{2}$ unit



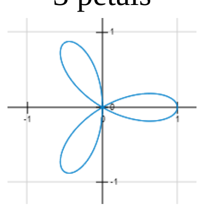
Below is the same graph on the Cartesian plane:



In general

- A rose with n positive and odd is of the form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ and has n petals of length a.

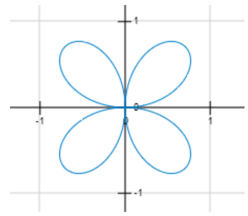
$r = \cos(3\theta)$
3 petals



- A rose with n positive and even is of the form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ and has $2n$ petals of length a

$$r = \sin(2\theta)$$

4 petals



Converting Equations

While many polar equations cannot be expressed nicely in Cartesian form (and vice versa), it can be beneficial to convert between the two forms, when possible. To do this we use the same relationships we used to convert points between coordinate systems.

Example 10: Convert the rectangular equation to a polar equation.

$y^2 - x^2 = 4$	Substitute $y = r\sin(\theta)$ and $x = r\cos(\theta)$
$(r\sin(\theta))^2 - (r\cos(\theta))^2 = 4$	Square
$r^2\sin^2(\theta) - r^2\cos^2(\theta) = 4$	Solve for r^2 by first factoring
$r^2(\sin^2(\theta) - \cos^2(\theta)) = 4$	Substitute $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
$r^2(-\cos(2\theta)) = 4$	Divide by $-\cos(2\theta)$
$r^2 = -\dfrac{4}{\cos(2\theta)}$	Reciprocal of cosine is secant
$r^2 = -4\sec(2\theta)$	Final answer

Example 11: Convert the polar equation to a rectangular equation

$r(3\cos(\theta) - 4\sin(\theta)) = 12$	Distribute
$3r\cos(\theta) - 4r\sin(\theta) = 12$	Substitute $r\cos(\theta) = x$ and $r\sin(\theta) = y$
$3x - 4y = 12$	Final answer

7.1 Polar Coordinates Practice

Graph each of the following points in polar coordinates.

1.	$(4, 30°)$	2.	$(5, 45°)$	3.	$(0, 37°)$	4.	$(0, 48°)$
5.	$(-6, 150°)$	6.	$(-5, 135°)$	7.	$(-8, 210°)$	8.	$(-5, 270°)$
9.	$(3, -30°)$	10.	$(6, -45°)$	11.	$(7, -315°)$	12.	$(5, -270°)$
13.	$(-3, -30°)$	14.	$(-6, -45°)$	15.	$(-3.2, 27°)$	16.	$(-6.8, 27°)$
17.	$\left(6, \frac{\pi}{4}\right)$	18.	$\left(5, \frac{\pi}{6}\right)$	19.	$\left(4, \frac{3\pi}{2}\right)$	20.	$\left(3, \frac{3\pi}{4}\right)$
21.	$\left(-6, \frac{\pi}{4}\right)$	22.	$\left(-5, \frac{\pi}{6}\right)$	23.	$\left(-4, -\frac{3\pi}{2}\right)$	24.	$\left(-3, -\frac{3\pi}{4}\right)$

Rewrite each of the following points given in rectangular coordinates into polar coordinates.

25.	$(4, 4)$	26.	$(5, 5)$	27.	$(0, 5)$	28.	$(0, -3)$
29.	$(4, 0)$	30.	$(-5, 0)$	31.	$(3, 3\sqrt{3})$	32.	$(-3, -3\sqrt{3})$
33.	$(\sqrt{3}, 1)$	34.	$(-\sqrt{3}, 1)$	35.	$(3\sqrt{3}, 3)$	36.	$(4\sqrt{3}, -4)$

Rewrite each of the following points given in polar coordinates into rectangular coordinates.

37.	$(4, 45°)$	38.	$(5, 60°)$	39.	$(0, 23°)$	40.	$(0, -34°)$
41.	$(-3, 45°)$	42.	$(-5, 30°)$	43.	$(6, -60°)$	44.	$(3, -120°)$
45.	$\left(10, \frac{\pi}{6}\right)$	46.	$\left(12, \frac{3\pi}{4}\right)$	47.	$\left(-5, \frac{5\pi}{6}\right)$	48.	$\left(-6, \frac{3\pi}{4}\right)$

Convert to a polar equation.

49.	$3x + 4y = 5$	50.	$5x + 3y = 4$	51.	$x = 5$	52.	$y = 4$
53.	$x^2 + y^2 = 36$	54.	$x^2 + y^2 = 16$	55.	$x^2 - 4y^2 = 4$	56.	$x^2 - 5y^2 = 5$

Convert to a rectangular equation.

57. $r = 5$

58. $r = 8$

59. $\theta = \dfrac{\pi}{4}$

60. $\theta = \dfrac{3\pi}{4}$

61. $r \sin\theta = 2$

62. $r \cos\theta = 5$

63. $r = 4\cos\theta$

64. $r = -3\cos\theta$

65. $r - r\sin\theta = 2$

66. $r + r\cos\theta = 8$

67. $r - 2\cos\theta = 3\sin\theta$

68. $r + 5\sin\theta = 7\cos\theta$

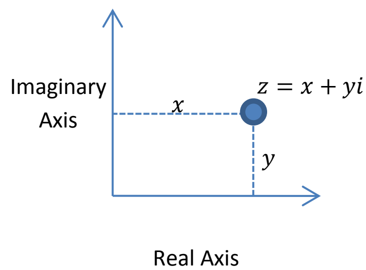
7.2 Polar Form of Complex Numbers

Imaginary Number i

The most basic complex number is i, defined to be $i = \sqrt{-1}$, commonly called an *imaginary number*. Any real multiple of i is also an imaginary number.

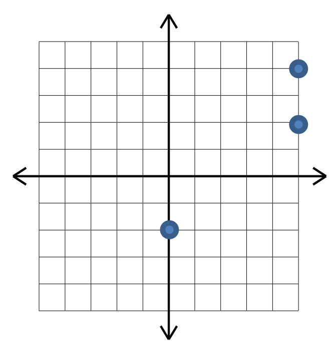
A complex number $z = x + yi$ is uniquely determined by the pair of real numbers (x, y). Thus we can identify the first and second elements of an ordered pair with the real and imaginary parts of z respectively. For example, $(3, -7)$ corresponds to $z = 3 - 7i$.

When each point in a coordinate plane is identified with a complex number in this way, the plane is called the *complex plane*. As the figure shows, the vertical or y-axis is called the *imaginary axis*, and the horizontal or x-axis is designated the *real axis*.



Example 1: Graph each of the following complex numbers, and then graph their sum

$$5 + 4i, -2i$$



The first point, $5 + 4i$, becomes $(5, 4)$

The second point, $-2i$ or $0 - 2i$, becomes $(0, -2)$

The sum is $5 + 4i + (-2i) = 5 + 2i$ which becomes $(5, 2)$

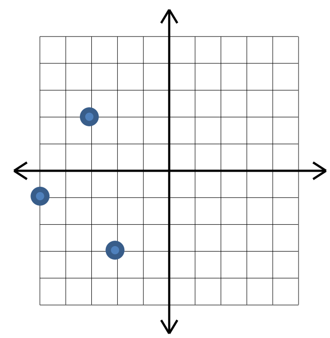
Example 2: Graph each of the following complex numbers, and graph their sum

$$-2-3i, -3+2i$$

The first point, $-2-3i$, becomes $(-2,-3)$

The second point, $-3+2i$, becomes $(-3,2)$

The sum is $(-2-3i)+(-3+2i)=-5-i$, or $(-5,-1)$



Polar Form of Complex Numbers

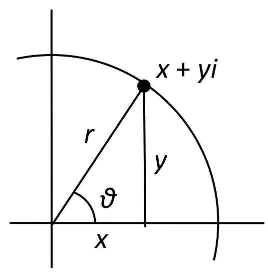
Remember, because the complex plane is analogous to the Cartesian (rectangular) plane that we can think of a complex number $z=x+yi$ as analogours to the Cartesian point (x,y) and recall how we converted from (x,y) to polar (r,θ) in the last section.

Bringing in all of our old rules we remember the following:

$$\cos(\theta)=\frac{x}{r} \qquad x=r\cos(\theta)$$

$$\sin(\theta)=\frac{y}{r} \qquad y=r\sin(\theta)$$

$$\tan(\theta)=\frac{y}{x} \qquad x^2+y^2=r^2$$



With this in mind, we can write $z=x+yi=rcos(\theta)+i\,r\sin(\theta)=r(\cos(\theta)+i\sin(\theta))$

Example 3: Find rectangular notation of $z=\sqrt{8}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$

$z=\sqrt{8}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$ Evaluate sine and cosine and simplify square root

$z=2\sqrt{2}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$ Distribute

$z=\sqrt{2}-i\sqrt{6}$ Final answer

Example 4: Find the polar notation of $1+i$

$1+i$	Find r
$r=\sqrt{(1)^2+(1)^2}=\sqrt{2}$	Find θ, note $(1,1)$ is in the first quadrant
$\cos(\theta)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$ $\theta=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$	Write in polar notation
$\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$	Final answer

In the 18th century, Leonhard Euler demonstrated a relationship between exponential and trigonometric functions that allows the use of complex numbers to greatly simplify some trigonometric calculations.

Polar Form of a Complex Number and Euler's Formula

The polar form of a complex number is $z=re^{i\theta}$ where Euler's formula holds:

$$re^{i\theta}=r\cos(\theta)+ir\sin(\theta)=r(\cos(\theta)+i\sin(\theta))$$

Similar to plotting a point in the coordinate system, we need r and θ to find the polar form of a complex number.

Example 5: Find the polar notation of -4.

-4	Write as a complex number
$-4+0i$	Note the point $(-4,0)$
$r=4$ $\theta=\pi$	This gives
$z=4e^{i\pi}$	Using Euler's Formula
$4(cos(\pi)+i\ sin(\pi)\ )$	Final answer

Example 6: Find the polar notation of $1 - \sqrt{3}i$

$1 - \sqrt{3}i$ Find r

$r = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$ Find θ, note $(1, -\sqrt{3})$ is in the fourth quadrant

$\cos(\theta) = \frac{1}{2}$ Write in polar notation

$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{5\pi}{3}$

$2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$ Final answer

The notation of $r(\cos(\theta) + i\sin(\theta))$ is often shortened to $r\ cis(\theta)$. Thus the above solution could equivalently be written

$$2\ cis\left(\frac{5\pi}{3}\right)$$

Multiplication and Division of Complex Numbers

Suppose we are given two complex numbers: $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$. The product of z_1 and z_2 is:

$$z_1z_2 = r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2))$$

Reorder factors to get:

$$z_1z_2 = r_1r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$$

FOIL the binomials to get:

$$z_1z_2 = r_1r_2(\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i^2\sin(\theta_1)\sin(\theta_2))$$

Since $i = \sqrt{-1}$ then $i^2 = -1$. Grouping together real and imaginary parts gives us:

$$z_1z_2 = r_1r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i\ (\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))]$$

Recall sum and difference identities: $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ and $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. This gives us

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Now the division of z_1 and z_2 is:

$$\frac{z_1}{z_2} = \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))}$$

We will rationalize the denominator by multiplying both numerator and denominator by the conjugate: $\cos(\theta_2) - i\sin(\theta_2)$. We obtain:

$$\frac{z_1}{z_2} = \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} \cdot \frac{(\cos(\theta_2) - i\sin(\theta_2))}{(\cos(\theta_2) - i\sin(\theta_2))}$$

FOIL out numerator and denominator. Notice the denominator is a sum and difference.

$$\frac{z_1}{z_2} = \frac{r_1(\cos(\theta_1)\cos(\theta_2) - i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - i^2\sin(\theta_1)\sin(\theta_2))}{r_2(\cos^2(\theta_2) - i^2\sin^2(\theta_2))}$$

Since $i = \sqrt{-1}$ then $i^2 = -1$. Grouping together real and imaginary parts gives us:

$$\frac{z_1}{z_2} = \frac{r_1[(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)) + i\,(\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2))]}{r_2(\sin^2(\theta_2) + \cos^2(\theta_2))}$$

In the denominator we know $\sin^2\alpha + \cos^2\alpha = 1$. In the numerator we can use the sum and difference formulas, $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ and $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Product and Quotient of Two Complex Numbers

Let $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$ be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example 7: Convert to polar and then multiply

$$(2+2i)(1-i)$$

Convert first factor to polar, first quadrant

$$r=\sqrt{(2)^2+(2)^2}=\sqrt{8}=2\sqrt{2}$$
$$\cos(\theta)=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$
$$\theta=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$$
$$z_1=2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$$

Convert the second factor, fourth quadrant

$$r=\sqrt{(1)^2+(1^2)}=\sqrt{2}$$
$$\cos(\theta)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$
$$\theta=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$$
$$z_2=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$$

Use product formula

$$z_1z_2=2\sqrt{2}\cdot\sqrt{2}\left[\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}-\frac{\pi}{4}\right)\right]$$

Simplify

$$z_1z_2=4(\cos(0)+i\sin(0))$$

Or alternatively

$$z_1z_2=4cis(0)$$

Final answer

Example 8: Convert to polar and then divide

$$\frac{2\sqrt{3}-2i}{-1+i\sqrt{3}}$$

Convert numerator, fourth quadrant

$$r=\sqrt{\left(2\sqrt{3}\right)^2+(-2)^2}=\sqrt{16}=4$$

$$\cos(\theta)=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$$

$$\theta=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=-30°$$

$$z_1=4(\cos(-30°)+i\sin(-30°))$$

Convert denominator, second quadrant

$$r=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$$

$$\cos(\theta)=-\frac{1}{2}$$

$$\theta=\cos^{-1}\left(-\frac{1}{2}\right)=120°$$

$$z_2=2(\cos(120°)+i\sin(120°))$$

Use quotient formula

$$\frac{z_1}{z_2}=\frac{4}{2}[\cos(-30°-120°)+i\sin(-30°-120°)]$$

Simplify

$$\frac{z_1}{z_2}=2(\cos(-150°)+i\sin(-150°))$$

Or alternatively

$$\frac{z_1}{z_2}=2cis(-150°)$$

Final answer

7.2 Polar Form of Complex Numbers

Graph each of the following complex numbers and then graph their sum.

1. $3 + 2i, 2 - 5i$
2. $4 + 3i, 3 - 4i$
3. $-5 + 3i, -2 - 3i$
4. $-4 + 2i, -3 - 4i$
5. $2 - 3i, -5 + 4i$
6. $3 - 2i, -5 + 5i$
7. $-2 - 5i, 5 + 3i$
8. $-3 - 4i, 6 + 3i$

Find rectangular notation

9. $3(\cos(30°) + i\sin(30°))$
10. $6(\cos(150°) + i\sin(150°))$
11. $10cis(270°)$
12. $12cis(-60°)$
13. $\sqrt{8}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$
14. $5\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$
15. $\sqrt{8}cis\left(\frac{5\pi}{4}\right)$
16. $\sqrt{8}cis\left(-\frac{\pi}{4}\right)$

Find polar notation

17. $1 - i$
18. $\sqrt{3} + i$
19. $10\sqrt{3} - 10i$
20. $-10\sqrt{3} + 10i$
21. -5
22. $-5i$

Convert to polar and then multiply or divide

23. $(1 - i)(2 + 2i)$
24. $\left(1 + i\sqrt{3}\right)(1 + i)$
25. $\left(2\sqrt{3} + 2i\right)(2i)$
26. $\left(2\sqrt{3} - 3i\right)(2i)$
27. $\dfrac{1 - i}{1 + i}$
28. $\dfrac{1 - i}{\sqrt{3} - i}$
29. $\dfrac{2\sqrt{3} - 2i}{1 + i\sqrt{3}}$
30. $\dfrac{3 - 3i\sqrt{3}}{\sqrt{3} - i}$

7.3 DeMoivre's Theorem

Powers of Complex Numbers

In this section we will study procedures for finding powers and roots of complex numbers. To begin consider the complex number (in polar form),

$$z = r(\cos(\theta) + i\sin(\theta))$$

If we multiply the left side of this equation by z and the right side by $r(\cos(\theta) + i\sin(\theta))$ yields:

$$\begin{aligned} z^2 &= r(\cos(\theta) + i\sin(\theta))r(\cos(\theta) + i\sin(\theta)) \\ &= r^2(\cos^2(\theta) + 2i\sin(\theta)\cos(\theta) - \sin^2(\theta)) \\ &= r^2(\cos^2(\theta) - \sin^2(\theta) + i \cdot 2\sin(\theta)\cos(\theta)) \\ &= r^2(\cos(2\theta) + i\sin(2\theta)) \end{aligned}$$

Continuing to multiply the left side by z and the right side by $r(\cos(\theta) + i\sin(\theta)$ yields the following pattern:

$$z = r(\cos(\theta) + i\sin(\theta))$$

$$z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$$

$$z^3 = r^3(\cos(3\theta) + i\sin(3\theta))$$

$$z^4 = r^4(\cos(4\theta) + i\sin(4\theta))$$

$$z^5 = r^5(\cos(5\theta) + i\sin(5\theta))$$

This pattern leads to the following important theorem, which is named after the French mathematician Abraham DeMoivre:

DeMoivre's Theorem:

If $z = r(\cos(\theta) + i\sin(\theta)$ is a complex number and n is a positive integer, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Example 1: Raise to the given power. Write the answer in polar notation

$$\left(2cis\left(\frac{\pi}{6}\right)\right)^8$$ Using DeMoivre's theorem

$$2^8\left(cis\left(8\cdot\frac{\pi}{6}\right)\right)$$ Simplify

$$256cis\left(\frac{4\pi}{3}\right)$$ Final answer

Example 2: Raise to the given power. Write the answer in polar notation

$$(-2+2i)^4$$ Convert to polar notation

$$r=\sqrt{(-2)^2+(2)^2}=\sqrt{8}=2\sqrt{2}$$ Give polar equation

$$\tan(\theta)=\frac{2}{-2}=-1$$

$$\theta=\tan^{-1}(-1)=\frac{3\pi}{4}$$

$$\left[2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)\right]^4$$ Using DeMoivre's Theorem

$$\left(2\sqrt{2}\right)^4\left(\cos\left(4\cdot\frac{3\pi}{4}\right)+i\sin\left(4\cdot\frac{3\pi}{4}\right)\right)$$ Simplify

$$64(\cos(3\pi)+i\sin(3\pi))$$ Final answer

Example 3: Raise to the given power. Write the answer in rectangular notation

$$(cis112.5°)^{24}$$ Using DeMoivre's Theorem

$$cis(24\cdot 112.5°)$$ Simplify

$$cis(2700°)$$ Expand

$$\cos(2700°)+i\sin(2700°)$$ Find conterminal angles

$$\cos(180°)+i\sin(180°)$$ Evaluate

$$-1+0i$$ Simplify

$$-1$$ Final answer

Example 4: Raise to the given power. Write the answer in rectangular notation

$$\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{6}}{2}\right)^9$$ Convert to polar notation

$$r = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{6}}{2}\right)^2} = \sqrt{2}$$ Give polar equation

$$\tan(\theta) = \frac{-\frac{\sqrt{6}}{2}}{\frac{\sqrt{2}}{2}} = -\sqrt{3}$$

$$\theta = \tan^{-1}(-\sqrt{3}) = 300°$$

$$[\sqrt{2}(\cos(300°) + i\sin(300°))]^9$$ Using DeMoivre's Theorem

$$(\sqrt{2})^9(\cos(9 \cdot 300) + i\sin(9 \cdot 300))$$ Simplify

$$16\sqrt{2}(\cos(2700°) + i\sin(2700))$$ Find conterminal angles

$$16\sqrt{2}(\cos(180°) + i\sin(180°))$$ Evaluate

$$16\sqrt{2}(-1 + 0i)$$ Simplify

$$-16\sqrt{2}$$ Final answer

Roots of Complex Numbers

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree n has n solutions in the complex number system. Hence, an equation like $z^6 = 1$ has six solutions, and in this particular case we can find the six solutions by factoring and using the quadratic formula:

$$z^6 = 1$$ Subtract 1

$$z^6 - 1 = 0$$ Factor, difference of squares

$$(z^3 - 1)(z^3 + 1) = 0$$ Factor sum and difference of cubes

$$(z - 1)(z^2 + z + 1)(z + 1)(z^2 - z + 1) = 0$$ Set factors equal to zero or use quadratic formula

$$z = \pm 1, \frac{-1 \pm i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}$$ Final answer

Each of these numbers is a sixth root of 1. In general, we define the n^{th} root of a complex number as follows:

Definition of n^{th} root of a complex number

The complex number $w = x + iy$ is an n^{th} root of the complex number z if

$$z = w^n = (x + iy)^n$$

To find a formula for an n^{th} root of a complex number, we let w be an n^{th} root of z, where $w = s(\cos(\alpha) + i\sin(\alpha))$ and $z = r(\cos(\theta) + i\sin(\theta))$. By DeMoivre's Theorem and the fact that $w^n = z$, we have

$$s^n(\cos(n\alpha) + i\sin(n\alpha)) = r(\cos(\theta) + i\sin(\theta))$$

Now, taking the absolute value of both sides of the equation, so $s^n = r$, which means

$$\cos(n\alpha) + i\sin(n\alpha) = \cos(\theta) + i\sin(\theta)$$

Since both sine and cosine have a period of 2π, these last two equations have solutions if and only if the angles differ by a multiple of 2π. Consequently, there must exist an integer k such that $n\alpha = \theta + 2\pi$ or $\alpha = \frac{\theta+2\pi}{n}$.

By substituting this value for α into the trigonometric form of w, we get the result stated in the following theorem.

n^{th} Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos(\theta) + i\sin(\theta))$ has exactly n distinct n^{th} roots given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

or

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

Where $k = 0,1,2,\ldots,n-1$

Example 5: Solve the equation for z

$$z^3 = i$$ Convert to polar notation

$$r = \sqrt{(0)^2 + (1)^2} = 1$$ Give polar equation

$$\tan(\theta) = \frac{1}{0} = undefined$$

$$\theta = \frac{\pi}{2}$$

$$z^3 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$ Using the root formula

$$w_k = \sqrt[3]{1}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right]$$

$$k = 0,1,2$$

$$w_0 = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$ Evaluate each value of k, start with $k = 0$

$$w_1 = \cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$$ Next evaluate $k = 1$

$$w_1 = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_2 = \cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)$$ Finally evaluate $k = 2$

$$w_2 = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - i = -i$$

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -i$$ List three solutions

Example 6: Solve the equation for z

$$z^2 = 1 + \sqrt{3}i$$

Polar notation

$$r = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

Give polar equation

$$\tan(\theta) = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\theta = \tan^{-1}(\sqrt{3}) = 60°$$

$$z^2 = 2(\cos(60°) + i\sin(60°))$$

Using root formula

$$w_k = \sqrt{2}\left[\cos\left(\frac{60° + 360k}{2}\right) + i\sin\left(\frac{60° + 360k}{2}\right)\right]$$

Simplify

$$w_k = \sqrt{2}[\cos(30° + 180k) + i\sin(30° + 180k)]$$

$$k = 0,1$$

$$w_0 = \sqrt{2}[\cos(30°) + i\sin(30°)] = \sqrt{2}\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

Evaluate $k = 0$

$$w_1 = \sqrt{2}[\cos(210°) + i\sin(210°)] = \sqrt{2}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

Evaluate $k = 1$

$$\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

List answers

Example 7: Find and graph the six roots of -1

$$-1$$

Covert to polar equation

$$-1 = 1(\cos(180°) + i\sin(180°))$$

Use root formula

$$w_k = 1^k\left[\cos\left(\frac{180° + 360k}{6}\right) + i\sin\left(\frac{180° + 360}{6}\right)\right]$$

Simplify

$$w_k = \cos(30° + 60k) + i\sin(30° + 60k)$$
$$k = 0, 1, 2, 3, 4, 5$$

$$w_0 = \cos(30°) + i\sin(30°) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Evaluate $k = 0$

$$w_1 = \cos(90°) + i\sin(90°) = 0 + i$$

Evaluate $k = 1$

$$w_2 = \cos(150°) + i\sin(150°) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Evaluate $k = 2$

$$w_3 = \cos(210°) + i\sin(210°) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Evaluate $k = 3$

$$w_4 = \cos(270°) + i\sin(270°) = 0 - i$$

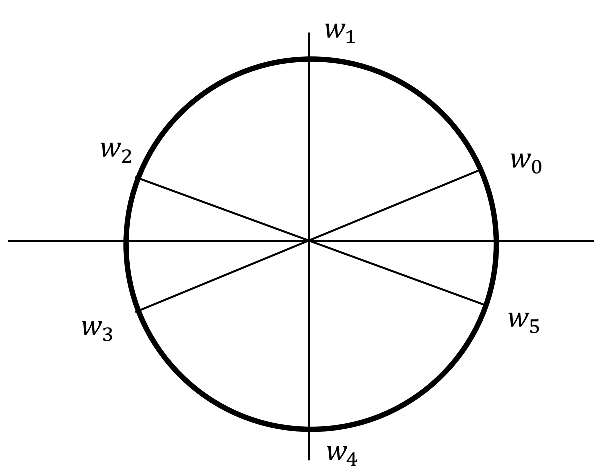
Evaluate $k = 4$

$$w_5 = \cos(330°) + i\sin(330°) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Evaluate $k = 5$

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i, i, -i$$

List solutions



Example 8: Find the solutions of the equation $z^4 + 81 = 0$

$$z^4 + 81 = 0$$ Subtract 81

$$z^4 = -81$$ Convert to Polar

$$z^4 = 81[\cos(180°) + i\sin(180°)]$$ Use root formula

$$w_k = \sqrt[4]{81}[\cos\left(\frac{180° + 360k}{4}\right) + i\sin\left(\frac{180° + 360k}{4}\right)$$ Simplify

$$w_k = 3(\cos(45° + 90k) + i\sin(45° + 90k))$$
$$k = 0, 1, 2, 3$$

$$w_0 = 3(\cos(45°) + i\sin(45°)) = 3\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$ Evaluate $k = 0$

$$w_1 = 3(\cos(135°) + i\sin(135°)) = 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$$ Evaluate $k = 1$

$$w_2 = 3(\cos(225°) + i\sin(225°)) = 3\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$ Evaluate $k = 2$

$$w_3 = 3(\cos(315°) + i\sin(315°)) = 3\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$ Evaluate $k = 3$

$$\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i, -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i, -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i, \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$ List solutions

7.3 DeMoivre's Theorem Practice

Raise each number to the given power. Write the answer in polar notation.

1. $\left(2cis\left(\frac{\pi}{3}\right)\right)^3$
2. $\left(3cis\left(\frac{\pi}{2}\right)\right)^4$
3. $\left(2cis\left(\frac{\pi}{6}\right)\right)^6$
4. $\left(2cis\left(\frac{\pi}{5}\right)\right)^5$
5. $(1+i)^6$
6. $(1-i)^6$

Raise each number to the given power. Write the answer in rectangular notation.

7. $\left(2cis(240°)\right)^4$
8. $\left(2cis(120°)\right)^4$
9. $\left(1+i\sqrt{3}\right)^4$
10. $\left(-\sqrt{3}+i\right)^6$
11. $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{10}$
12. $\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^{12}$
13. $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{12}$
14. $\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{14}$

Solve the following equations for z

15. $z^2 = i$
16. $z^2 = -i$
17. $z^2 = 2\sqrt{2} - 2i\sqrt{2}$
18. $z^2 = 2\sqrt{2} + 2i\sqrt{2}$
19. $z^2 = -1 + i\sqrt{3}$
20. $z^2 = -\sqrt{3} - i$
21. $z^3 = i$
22. $z^3 = 68.4321$
23. Find and graph the fourth roots of 16
24. Find and graph the fourth roots of i
25. Find and graph the fifth roots of -1
26. Find and graph the sixth roots of 1
27. Find the fourth roots of $-8+8i\sqrt{3}$
28. Find the cube roots of $-64i$
29. Find the cube roots of $2\sqrt{3}-2i$
30. Find the cube roots of $1-i\sqrt{3}$

Find all complex solutions of the following equations

31. $z^3 = 1$

32. $z^5 - 1 = 0$

33. $z^5 + 1 = 0$

34. $z^4 + i = 0$

35. $z^5 + \sqrt{3} + i = 0$

36. $z^6 + 1 = 0$

Chapter 8

Sequences and Series

8.1 Sequences

Sequences

Sequence: Infinitely-long ordered list of numbers. Equivalently, a function whose domain is the natural numbers (1, 2, 3…)

We usually try to write sequences as a general n-th term, which we call a_n.

Example 1: Write the first five terms of the sequence $a_n = 3 + 5n$, assuming the sequence begins with a_1.

$$a_n = 3 + 5n \qquad \text{Replace } n \text{ with } 1, 2, 3, 4, 5$$

$$\begin{aligned} a_1 &= 3 + 5(1) = 8 \\ a_2 &= 3 + 5(2) = 13 \\ a_3 &= 3 + 5(3) = 18 \\ a_4 &= 3 + 5(4) = 23 \\ a_5 &= 3 + 5(5) = 28 \end{aligned} \qquad \text{List first five terms}$$

$$8, 13, 18, 23, 28 \qquad \text{Final answer}$$

Factorials and Sequences

Sometimes a sequence will use what is called a factorial. This is defined below.

Factorial: $n!$ Refers to a natural number n multiplied by every number below it.

$$n! = n \cdot (n-1) \cdot (n-2) \dots 3 \cdot 2 \cdot 1$$

Also, $0! = 1$

Example 2: Simplify the expression $\frac{9!}{6!}$

$$\frac{9!}{6!} \qquad \text{Write out expression longhand}$$

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \qquad \text{Divide out common factor (1-6)}$$

$$9 \cdot 8 \cdot 7 \qquad \text{Multiply}$$

$$504 \qquad \text{Final answer}$$

Example 3: Simplify the expression $\frac{(n+3)!}{(n+1)!}$

$$\frac{(n+3)!}{(n+1)!}$$ Expand each, multiplying by one less each time

$$\frac{(n+3)(n+2)(n+1)n\ldots3\cdot2\cdot1}{(n+1)n(n-1)(n-2)\ldots3\cdot2\cdot1}$$ Divide out common factors

$$(n+3)(n+2)$$ FOIL

$$n^2+5n+6$$ Final answer

Now we will consider a sequence which uses a factorial in its definition.

Example 4: Write the first five terms of the sequence $a_n=(-1)^n\frac{(3+n)}{n!}$

$$a_n=(-1)^n\frac{(3+n)}{n!}$$ Replace n with $1,2,3,4,5$

$$a_1=(-1)^1\frac{3+1}{1!}=-\frac{4}{1}=-4$$
$$a_2=(-1)^2\frac{3+2}{2!}=\frac{5}{2}$$
$$a_3=(-1)^3\frac{3+3}{3!}=-\frac{6}{6}=-1$$
$$a_4=(-1)^4\frac{3+4}{4!}=\frac{7}{24}$$
$$a_5=(-1)^5\frac{3+5}{5!}=-\frac{8}{120}=-\frac{1}{15}$$ List first five terms

$$-4,\frac{5}{2},-1,\frac{7}{24},-\frac{1}{15}$$ Final answer

Recursive Sequences

Recursive Sequence: Sequence in which we define each successive term using previous terms

Example 5: Write the first five terms of the recursive sequence $a_k = 1.5a_{k-1}$ where $a_1 = 4$

$a_1 = 4$	Using our definition for a_k where a_{k-1} is 4
$a_2 = 1.5(4) = 6$	Use the definition again where a_{k-1} is 6
$a_3 = 1.5(6) = 9$	Continue to find a_4 and a_5
$a_4 = 1.5(9) = 13.5$ $a_5 = 1.5(13.5) = 20.25$	List first five terms
$4, 6, 9, 13.5, 20.25$	Final answer

Example 6: Write the first five terms of the recursive sequence $a_k = a_{k-1} + a_{k-2}$ where $a_0 = 0$ and $a_1 = 1$

$a_0 = 0, a_1 = 1$	To find a_2 we add $a_1 + a_0$
$a_2 = 1 + 0 = 1$	To find a_3 we add $a_2 + a_1$
$a_3 = 1 + 1 = 2$	Continue to find a_4 and a_5
$a_4 = 2 + 1 = 3$ $a_5 = 3 + 2 = 5$	List first five terms
$0, 1, 1, 2, 3, 5$	Final answer

We may be asked to find the n-th term of a sequence. Here we will need to look for patterns such as a difference between terms, a quotient between terms, factorials, or exponents.

Example 7: Find the n-th term of the sequence: $2, 9, 16, 23, 30$ ...

$2, 9, 16, 23, 30$	Notice terms increase by a constant 7. Consider $7n$
$7n$	If we replace $n = 1$ we get
$7(1) = 7$	Our first term is 2, we must subtract 5
$a_n = 7n - 5$	Check with the second point
$a_2 = 7(2) - 5 = 9$	It works!
$a_n = 7n - 5$	Final answer

Example 8: Find the n-th term of the sequence $\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243} \dots$

We will consider the numerators and denominators separately to identify a pattern.

Numerators: $1, 4, 9, 16, 25$ ...	Notice this is the list of squares
Numerators: n^2	A quick check verifies this works for all numerators
Denominators: $3, 9, 27, 81, 243$ ...	Notice we are multiplying by 3. Try 3^n
Denominators: 3^n	A quick check verifies this works for all denominators
$a_n = \frac{n^2}{3^n}$	Final answer

8.1 Sequences Practice

Write the first five terms of the sequences, assuming it starts at $n = 1$

1. $a_n = 2n + 1$
2. $a_n = 4n - 3$
3. $a_n = 2^n$
4. $a_n = \left(\frac{1}{2}\right)^n$
5. $a_n = (-2)^n$
6. $a_n = \left(-\frac{1}{2}\right)^n$
7. $a_n = \dfrac{1 + (-1)^n}{n}$
8. $a_n = \dfrac{n}{n+1}$
9. $a_n = 3 - \dfrac{1}{2^n}$
10. $a_n = \dfrac{3^n}{4^n}$
11. $a_n = \dfrac{1}{n^{3/2}}$
12. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$
13. $a_n = \dfrac{3^n}{n!}$
14. $a_n = \dfrac{n!}{n}$
15. $a_n = \dfrac{(-1)^n}{n^2}$
16. $a_n = (-1)^n \left(\dfrac{n}{n+1}\right)$
17. $a_{k+1} = 2(a_k - 1)$ where $a_1 = 3$
18. $a_k = a_{k-1}\left(\frac{k+1}{2}\right)$ where $a_1 = 4$

Write an expression for the n-th term of the sequence

19. $1, 4, 7, 10, 13 \ldots$
20. $3, 7, 11, 15, 19 \ldots$
21. $0, 3, 8, 15, 24 \ldots$
22. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25} \ldots$
23. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16} \ldots$
24. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81} \ldots$
25. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3} \ldots$
26. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8} \ldots$
27. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \ldots$
28. $2, -4, 6, -8, 10 \ldots$
29. $1, -1, 1, -1, 1 \ldots$
30. $2, \frac{2^2}{2}, \frac{2^3}{6}, \frac{2^4}{24}, \frac{2^5}{120} \ldots$

Simplify the expression

31. $\dfrac{4!}{6!}$
32. $\dfrac{25!}{23!}$
33. $\dfrac{(n+2)!}{n!}$
34. $\dfrac{(n+1)!}{n!}$
35. $\dfrac{(2n-1)!}{(2n+1)!}$
36. $\dfrac{(2n+2)!}{(2n)!}$

8.2 Series

Series

A *Series* is a sum of the numbers in a sequence.

Summation Notation: Shorthand for the terms in a sequence expressed as a sum with the Greek letter $\sum$

$$\sum_{\text{Starting Index Value}}^{\text{Ending Index Value}} \text{General Term}$$

To evaluate the sum we replace the variable in the general term with each index value between the starting and ending index value. Then sum all the terms to get a final solution.

Example 1: Find the sum.

$$\sum_{i=1}^{5} 2i+3$$ Evaluate each term with $i = 1, 2, 3, 4, 5$

$$2(1)+3=5$$
$$2(2)+3=7$$
$$2(3)+3=9$$
$$2(4)+3=11$$
$$2(5)+3=13$$ Add them up

$$5+7+9+11+13=45$$ Final answer

Example 2: Find the sum.

$$\sum_{k=2}^{6} \frac{(-1)^k 2^{k-1}}{k!}$$

Evaluate each term with $k = 2, 3, 4, 5, 6$

$$\frac{(-1)^2 2^{2-1}}{2!} = \frac{(1)(2^1)}{2} = 1$$

$$\frac{(-1)^3 2^{3-1}}{3!} = \frac{(-1)(2^2)}{3 \cdot 2} = -\frac{4}{6} = -\frac{2}{3}$$

$$\frac{(-1)^4 2^{4-1}}{4!} = \frac{(1)(2^3)}{4 \cdot 3 \cdot 2} = \frac{8}{24} = \frac{1}{3}$$

$$\frac{(-1)^5 2^{5-1}}{5!} = \frac{(-1)(2^4)}{5 \cdot 4 \cdot 3 \cdot 2} = -\frac{16}{120} = -\frac{2}{15}$$

$$\frac{(-1)^6 2^{6-1}}{6!} = \frac{(1)(2^5)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{32}{720} = \frac{2}{45}$$

Add

$$1 + \left(-\frac{2}{3}\right) + \frac{1}{3} + \left(-\frac{2}{15}\right) + \frac{2}{45} = \frac{26}{45}$$

Final answer

Example 3: Rewrite this series in summation notation: $\frac{5}{2} + \frac{5}{6} + \frac{5}{24} + \frac{5}{120} + \frac{5}{720} + \frac{5}{5040}$

As with sequences, we want to look for patterns in the numbers. The numerator clearly is always 5. We will look at the denominator in search of a pattern

$2, 6, 24, 120, 720, 5040$ These values are factorial values. Test $n!$ With $n = 2, 3, 4, 5, 6, 7$

$$\sum_{n=2}^{7} \frac{5}{n!}$$

Final answer

Example 4: Rewrite this series in summation notation: $\frac{4}{3} - \frac{5}{9} + \frac{6}{27} - \frac{7}{81} + \frac{8}{243}$

Notice the signs alternate. When this happens we often use $(-1)^n$ or $(-1)^{n+1}$ as a factor depending on which gives us the correct sign. Now we can look at the fractions. We begin by separating the numerator and denominator to make patterns easier to identify:

$4, 5, 6, 7, 8$	As each is increasing by 1 we try $1n$ or n.
n	To start with the 1st term at 4 we need to add 3
$n + 3$	Now we look at the denominators
$3, 9, 27, 81, 243$	These are multiples of 3
3^n	Put all three parts together
$\sum_{n=1}^{5} (-1)^{n+1} \frac{n+3}{3^n}$	Final answer

With any series, we have more than one possible answer. We can use any letter we like for the index variable. More to the point, if you chose a starting index value of 4, this would have worked as well:

$$\sum_{j=4}^{8} (-1)^j \frac{j}{3^{j-3}}$$

8.2 Series Practice

Find the given sum.

1. $\sum_{i=1}^{5} 2i+1$

2. $\sum_{k=1}^{6} 3k-1$

3. $\sum_{k=1}^{4} 10$

4. $\sum_{k=1}^{5} 6$

5. $\sum_{j=0}^{4} j^2$

6. $\sum_{i=0}^{4} 3i^2$

7. $\sum_{k=0}^{3} \frac{1}{k^2+1}$

8. $\sum_{j=3}^{5} \frac{1}{j}$

9. $\sum_{i=1}^{4} [(1-i)^2+(i+i)^3]$

10. $\sum_{k=2}^{5} (k+1)(k-3)$

11. $\sum_{i=0}^{4} 9+2i$

12. $\sum_{j=0}^{4} (-2)^j$

13. $\sum_{k=0}^{4} \frac{(-1)^k}{k+1}$

14. $\sum_{k=0}^{4} \frac{(-1)^k}{k!}$

Rewrite in summation notation

15. $\frac{1}{3(1)}+\frac{1}{3(2)}+\frac{1}{3(3)}+\cdots+\frac{1}{3(9)}$

16. $\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+\cdots+\frac{5}{1+15}$

17. $3-9+27-81+243-729$

18. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots-\frac{1}{128}$

19. $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots-\frac{1}{20^2}$

20. $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{10\cdot 12}$

21. $\frac{1}{4}+\frac{3}{8}+\frac{7}{16}+\frac{15}{32}+\frac{31}{64}$

22. $\frac{1}{2}+\frac{2}{4}+\frac{6}{8}+\frac{24}{16}+\frac{120}{32}+\frac{720}{64}$

23. $\left[2\left(\frac{1}{8}\right)+3\right]+\left[2\left(\frac{2}{8}\right)+3\right]+\cdots+\left[2\left(\frac{8}{8}\right)+3\right]$

24. $\left[1-\left(\frac{1}{6}\right)^2\right]+\left[1-\left(\frac{2}{6}\right)^2\right]+\cdots+\left[1-\left(\frac{6}{6}\right)^2\right]$

8.3 Arithmetic Series

An *Arithmetic Series* is a series that increases linearly. To find each successive term, we take the previous term and add d, the *common difference*

Formulas for Arithmetic Series

Recursive formula:

$$a_n = a_{n-1} + d$$

Where a_n is the n-th term, a_{n-1} is the previous term and d is the common difference

Explicit formula:

$$a_n = a_1 + d(n-1)$$

Where a_n is the n-th term, a_1 is the first term, d is the common difference, and n is the term number.

n-th partial sum: sum of the first n numbers in a sequence

$$S_n = \sum_{i=1}^{n} a_i$$

Partial sums of arithmetic series: The n-th partial sum of an arithmetic series looks like…

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Where a_n is the n-th term, a_1 is the first term,n is the term number, and S_n is the sum.

Example 1: Add up every integer from 1 to 100: $1 + 2 + 3 + \cdots + 99 + 100$

$1 + 2 + 3 + \cdots + 99 + 100$	Note the difference between each term is 1
$d = 1, a_1 = 1, a_{100} = 100$	Plug values into formula
$S_{100} = \frac{100}{2}(1 + 100)$	Evaluate
5050	Final answer

Note: According to the eventual college professor Wolfgang Sartorius, renowned mathematician Carl Friedrich Gauss was given this very problem in primary school at age 9 and solved it – using the method above, which he derived himself – in seconds!

Example 2: Find d, a_n, and S_n: $2, 5, 8, 11, \ldots$ $(n = 50)$

$2, 5, 8, 11, \ldots$	Subtracting terms we find d
$d = 5 - 2 = 3$ and $d = 8 - 5 = 3$	Using the explicit formula to find a_{50}
$a_{50} = 2 + 3(50 - 1)$	Evaluate
$a_{50} = 149$	Use the partial sum formula
$S_{50} = \dfrac{50}{2}(2 + 149)$	Evaluate
$S_{50} = 3775$	Final answer

Example 3: Find d, a_n and S_n: $25, 21, 17, 13, \ldots$ $(n = 10)$

$25, 21, 17, 13, \ldots$	Subtracting terms we find d
$d = 21 - 25 = -4$ and $d = 17 - 21 = -4$	Using the explicit formula to find a_{10}
$a_{10} = 25 - 4(10 - 1)$	Evaluate
$a_{10} = -11$	Use the partial sum formula
$S_{10} = \dfrac{10}{2}(25 - 11)$	Evaluate
$S_{10} = 70$	Final answer

Example 4: Find d, a_n, and S_n: $1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \ldots$ $(n = 17)$

$1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \ldots$	Subtracting terms we find d
$d = \frac{9}{8} - 1 = \frac{9}{8} - \frac{8}{8} = \frac{1}{8}$ $d = \frac{5}{4} - \frac{9}{8} = \frac{10}{8} - \frac{9}{8} = \frac{1}{8}$	Using the explicit formula to find a_{17}
$a_{17} = 1 + \frac{1}{8}(17 - 1)$	Evaluate
$a_{17} = 3$	Use partial sum formula
$S_{17} = \frac{17}{2}(1 + 3)$	Evaluate
$S_{17} = 34$	Final answer

Example 5: Given $a_1 = -10$, $d = 3$, and $a_n = 14$, find n and S_n

We can use the explicit formula which uses given information with one unknown

$14 = -10 + 3(n - 1)$	Distribute
$14 = -10 + 3n - 3$	Combine like terms
$14 = 3n - 13$	Add 13
$27 = 3n$	Divide by 3
$9 = n$	Use partial sum formula
$S_9 = \frac{9}{2}(-10 + 14)$	Evaluate
$S_9 = 18$	Final answer

Example 6: Given $a_9 = 31$ and $S_9 = 135$, find d and a_1

The partial sum formula uses the given information with one unknown

$135 = \frac{9}{2}(a_1 + 31)$	Multiply both sides be $\frac{2}{9}$ to clear the fraction
$30 = a_1 + 31$	Subtract 31
$-1 = a_1$	Use the explicit formula to solve for d
$31 = -1 + d(9 - 1)$	Simplify
$31 = -1 + 8d$	Add 1
$32 = 8d$	Divide by 8
$4 = d$	Final answer

8.3 Arithmetic Series Practice

In each of the following exercises, the sequence is an arithmetic progression. In each case find d, a_n and S_n

1. $2, 4, 6, 8, \ldots (n = 50)$
2. $5, 10, 15, 20, \ldots (n = 15)$
3. $1, 3, 5, 7, \ldots (n = 10)$
4. $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots (n = 8)$
5. $13, 10, 7, 4, \ldots (n = 10)$
6. $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0, \ldots (n = 10)$
7. $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots (n = 12)$
8. $1, 1.1, 1.2, 1.3, \ldots (n = 11)$
9. $0.75, 0.7, 0.65, 0.6, \ldots (n = 20)$

For each of the following arithmetic progressions find the requested parts:

10. Given $a_1 = 3, a_n = 48$ and $d = 5$, find n and S_n
11. Given $a_{12} = -59$ and $d = -5$, find a_1 and S_{12}
12. Given $S_{10} = -55$ and $a_1 = 8$, find d and a_{10}
13. Given $a_8 = 25$ and $S_8 = 88$, find a_1 and d
14. Given $d = 4, a_1 = -3$, find a_{12} and S_{12}
15. Given $S_n = 116, d = 3$ and $a_1 = 4$, find n and a_n
16. Given $d = -3, a_n = 2$ and $S_n = 155$, find n and a_1
17. Given $a_1 = 12$ and $a_8 = -9$, find d and S_8
18. Given $a_n = 3, a_1 = 31$ and $S_n = 136$, find n and d
19. Given $d = 5$ and $S_{10} = 315$, find a_1 and a_{10}
20. Given $a_1 = 2$ and $a_7 = 6$, find d and S_7
21. Given $a_1 = 0$ and $S_5 = 50$, find d and a_5
22. Given $a_{12} = 29$ and $S_{12} = 150$, find a_1 and d

8.4 Geometric Series

A *Geometric Series* is a series that increases exponentially. Each successive term comes from the previous term multiplied by r, the *common multiplier*

Formulas for Geometric Series

Recursive Formula

$$a_n = ra_{n-1}$$

Where a_n is the n-th term, r is the common multiplier, and a_{n-1} is the previous term

Explicit Formula

$$a_n = r^{n-1}a_1$$

Where a_n is the n-th term, r is the common multiplier, n is the term number, and a_1 is the first term

Partial Sums of Geometric Series

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Where S_n is the partial sum, a_1 is the first term, r is the common multiplier, n is the term number

Infinite Sums of Geometric Series

When $-1 < r < 1$, it turns out we can add up every term of a geometric series, even to infinity!

$$S = \frac{a_1}{1-r}$$

You may also see S written as S_∞. When $r < -1$ or $r > 1$ the infinite series cannot be determined.

Example 1: Find the tenth term of the geometric sequence: $2, 6, 18, 54$ ...

$2, 6, 18, 54, \ldots$ First find r by dividing terms

$r = \frac{6}{2} = 3$ Use the explicit formula for a_{10}

$r = \frac{18}{6} = 3$

$a_{10} = 3^{10-1} \cdot 2$ Evaluate

$a_{10} = 39366$ Final answer

Example 2: Find the ninth term of the geometric sequence: $256, -128, 64, -32, \ldots$

$256, -128, 64, -32, \ldots$ First find r by dividing terms

$r = \frac{-128}{256} = -\frac{1}{2}$ Use the explicit formula for a_9

$r = \frac{64}{-128} = -\frac{1}{2}$

$a_9 = \left(-\frac{1}{2}\right)^{9-1} \cdot 256$ Evaluate

$a_9 = 1$ Final answer

Example 3: Find the sum of the first 8 terms of $-1, 4, -16, 64, \ldots$

$-1, 4, -16, 64$ First find r by dividing terms

$r = \frac{4}{-1} = -4$ Use the partial sum formula

$r = \frac{-16}{4} = -4$

$S_8 = \frac{-1(1-(-4)^8)}{1-(-4)}$ Evaluate

$S_8 = 13107$ Final answer

Example 4: Find the sum of the geometric sequence $5, -\frac{5}{7}, \frac{5}{49}, -\frac{5}{343}, \ldots$

$$5, -\frac{5}{7}, \frac{5}{49}, -\frac{5}{343}, \ldots$$ First find r by dividing terms

$$r = \frac{-\frac{5}{7}}{5} = \left(-\frac{5}{7}\right)\left(\frac{1}{5}\right) = -\frac{1}{7}$$ Use the infinite sum formula

$$r = \frac{\frac{5}{49}}{-\frac{5}{7}} = \left(\frac{5}{49}\right)\left(-\frac{7}{5}\right) = -\frac{1}{7}$$

$$S = \frac{5}{1 - \left(-\frac{1}{7}\right)}$$ Multiply each term, top and bottom, by 7

$$S = \frac{35}{7 + 1}$$ Simplify

$$S = \frac{35}{8} = 4.375$$ Final answer

Example 5: Find the sum of this geometric sequence: $1000, 900, 810, 729, \ldots$

$$1000, 900, 810, 729, \ldots$$ First find r by dividing terms

$$r = \frac{900}{1000} = 0.9$$ Use the infinite sum formula

$$r = \frac{810}{900} = 0.9$$

$$S = \frac{1000}{1 - 0.9}$$ Simplify

$$S = \frac{1000}{0.1}$$ Divide

$$S = 10000$$ Final answer

Example 6: Given $a_1 = 6$ and $a_4 = \frac{16}{9}$, find r and S_4

$a_1 = 6, a_4 = \frac{16}{9}$	Use the explicit formula to find r
$\frac{16}{9} = r^{4-1} \cdot 6$	Simplify
$\frac{16}{9} = 6r^3$	Divide by 6 (multiply by $\frac{1}{6}$)
$\frac{8}{27} = r^3$	Cube root
$\frac{2}{3} = r$	Use partial sum formula
$S_4 = \frac{6\left(1-\left(\frac{2}{3}\right)^4\right)}{1-\frac{2}{3}}$	Exponent
$S_4 = \frac{6\left(1-\frac{16}{81}\right)}{1-\frac{2}{3}}$	Distribute
$S_4 = \frac{6-\frac{32}{27}}{1-\frac{2}{3}}$	Multiply each term, top and bottom, by 27
$S_4 = \frac{162-32}{27-18}$	Simplify
$S_4 = \frac{130}{9}$	Final answer

Example 7: Given $a_1 = \frac{1}{3}$ and $a_6 = -\frac{81}{32}$ find r and S_∞.

$a_1 = \frac{1}{3}, a_6 = -\frac{81}{32}$	Using the explicit formula, find r
$-\frac{81}{32} = r^{6-1} \cdot \left(\frac{1}{3}\right)$	Simplify
$-\frac{81}{32} = \frac{1}{3} r^5$	Multiply by 3
$-\frac{243}{32} = r^5$	Fifth root
$-\frac{3}{2} = r$	Notice $r < -1$

For an infinite series to have a sum, it is required that $-1 < r < 1$. This means the series will not sum to a specific value.

8.4 Geometric Series Practice

Find the indicated term in each of the following geometric progressions

1. The tenth term of $3, 6, 12, \ldots$

2. The eight term of $15, 5, \frac{5}{3}, \ldots$

3. The seventh term of $-2, 8, -32, \ldots$

4. The ninth term of $9, 6, 4, \ldots$

Find the sum of each of these finite geometric progressions in exercises 5-11:

5. The ten terms of exercise 1.

6. The eight terms of exercises 2.

7. The seven terms of exercise 3.

8. The nine terms of exercise 4.

9. Twenty terms of $1, -1, 1, \ldots$

10. Nineteen terms of $1, -1, 1, \ldots$

11. Twelve terms of $5, -1, 0.2, \ldots$

12. Find the sum of each infinite geometric progression

    a. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

    b. $-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$

    c. $18 + 12 + 8 + \frac{16}{3} + \cdots$

    d. $5 - 1 + \frac{1}{5} - \frac{1}{25} + \cdots$

13. Given $a_1 = 36$ and $a_5 = \frac{4}{9}$, find r and S_5

14. Given $a_6 = -6250$ and $r = -5$, find a_1 and S_6

15. Given $a_n = 16, r = -2$ and $S_n = 10\frac{5}{8}$, find a_1 and n

16. Given $a_1 = 2$ and $a_6 = 486$, find r and S_6

17. Given $r = 2$ and $S_8 = 127.5$, find a_1 and a_8

18. Given $a_1 = 16$ and $r = \frac{1}{2}$, find a_6 and S_∞

19. Given $a_1 = 10$ and $a_4 = -2.16$, find r and S_∞

20. Given $a_1 = 24$ and $S_\infty = 96$, find r and a_3

21. If the first swing of a pendulum is 12 inches and each swing is 0.7 of the previous swing, how far does the pendulum travel before coming to rest? To the nearest inch, how far will it have traveled at the end of the fourth swing?

22. One-fourth of the air in a given container is removed by each stroke of a vacuum pump. What fractional part of the original amount of air has been removed after four strokes of the pump?

8.5 Mathematical Induction

Definitions and Properties

Proof: A deductive argument that establishes the verity of a mathematical statement

Proof by Induction: A kind of proof for use on statements taking place over the natural numbers. To wit, we use proof by induction on problems where there exists a different case for each natural number.

Proof by induction has three steps:

1. *Base Case*: Prove that the statement is true for the first case (usually $n = 0$ or $n = 1$)
2. *Assumption Step:* if the $n = k$ case is true
3. *Inductive Step:* then prove that $n = (k + 1)$ is true.

Examples:

Example 1: Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

This can be also be written using summation notation as: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
We begin with the base case: $n = 1$. Substituting 1 for *i*, we get 1 on the right hand side. On the left side substituting 1 for n, we have

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

So$1 = 1$. This takes care of the base case.

For step 2, assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true.

Now let's look at the $n = k + 1$ case for both sides of that statement, this is what we want to show: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ is true. We want to show that these two statements equal each other. When doing this, we use the $n = k$ case.

The 1st term comes from the $n = k$ left hand side. The second term comes from substituting $k + 1$ in for the *i* in the summation expression. We want this to equal the left hand side of the $k + 1$ summation expression.

$$\frac{k(k+1)}{2} + (k+1)$$ Get a common denominator.

$$\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$ Distribute

$$\frac{k^2+k+2k+2}{2}$$ Combine like terms

$$\frac{k^2+3k+2}{2}$$ Factor

$$\frac{(k+1)(k+2)}{2}$$ We have now proved that the $n = k + 1$ case is true.

Example 2: Prove that $3^1 + 3^2 + 3^3 + \cdots + 3^n = \frac{3}{2}(3^n - 1)$

$$\sum_{i=1}^{n} 3^i = \frac{3}{2}(3^n - 1)$$

We begin with the base case: $n = 1$. On the left-hand side, we have $3^1 = 3$. On the right side we have

$$\frac{3}{2}(3^1 - 1) = \frac{3}{2}(2) = 3$$

So $3 = 3$. This takes care of the base case.

For the next step, assume $n = k$ is true.

$$\sum_{i=1}^{k} 3^i = \frac{3}{2}(3^k - 1)$$

Now prove that $n = k + 1$ is true, this is what we want to show:

$$\sum_{i=1}^{k+1} 3^i = \frac{3}{2}(3^{k+1} - 1)$$

We want to show that these two statements equal each other.

$$\frac{3}{2}(3^k - 1) + 3^{k+1}$$ Distribute $\frac{3}{2}$. Rewrite 3^{k+1} as $3^k(3)$

$$\left(\frac{3}{2}\right)3^k - \frac{3}{2} + 3^k(3)$$ Reorder terms

$3^k\left(\frac{3}{2}\right)+3^k(3)-\frac{3}{2}$ Factor the 3^k out of first two terms

$\left(\frac{3}{2}+3\right)3^k-\frac{3}{2}$ Simplify

$\frac{9}{2}3^k-\frac{3}{2}$ Factor out $\frac{3}{2}$

$\frac{3}{2}(3\cdot 3^k-1)$ Simplify exponents

$\frac{3}{2}(3^{k+1}-1)$ We have now proved that the $n=k+1$ case is true.

Example 3: Prove that $\sum_{i=1}^{n}\left(\frac{3}{2}\right)^i=3\left(\frac{3}{2}\right)^n-3$

We begin with the base case, $n=1$. On the left-hand side, we have

$$\sum_{i=1}^{1}\left(\frac{3}{2}\right)^i=\left(\frac{3}{2}\right)^1=\frac{3}{2}$$

On the right hand side we have

$$3\left(\frac{3}{2}\right)^1-3=3\left(\frac{3}{2}\right)-3=\frac{9}{2}-\frac{6}{2}=\frac{3}{2}$$

So $\frac{3}{2}=\frac{3}{2}$. This takes care of the base case.

For the next step, assume $n=k$ is true.

$\sum_{i=1}^{k}\left(\frac{3}{2}\right)^i=3\left(\frac{3}{2}\right)^k-3$

Now prove that $n=k+1$ is true, this is what we want to show:

$$\sum_{i=1}^{k+1}\left(\frac{3}{2}\right)^i=3\left(\frac{3}{2}\right)^{k+1}-3$$

We want to show that these two statements equal each other.

$3\left(\frac{3}{2}\right)^k - 3 + \left(\frac{3}{2}\right)^{k+1}$	Rewrite $\left(\frac{3}{2}\right)^{k+1}$ as $\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^k$ and reorder terms
$3\left(\frac{3}{2}\right)^k + \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^k - 3$	Factor out $\left(\frac{3}{2}\right)^k$
$\left(3 + \frac{3}{2}\right)\left(\frac{3}{2}\right)^k - 3$	Simplify
$\left(\frac{9}{2}\right)\left(\frac{3}{2}\right)^k - 3$	Factor $\frac{9}{2} = 3 \cdot \left(\frac{3}{2}\right)$
$3 \cdot \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^k - 3$	Simplify exponents
$3\left(\frac{3}{2}\right)^{k+1} - 3$	We have now proved that the $n = k + 1$ case is true.

8.5 Mathematical Induction Practice

Use Mathematical Induction to prove the following:

1. $$\sum_{p=1}^{n} \frac{1}{2^p} = \frac{2^n - 1}{2^n}$$

2. $$\sum_{p=1}^{n} \frac{1}{3^p} = \frac{3^n - 1}{2 \cdot 3^n}$$

3. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

4. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

5. $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

6. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

7. $$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$$

8. $$1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

9. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

10. $$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

11. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$

12. $$2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + (n+1)(n+4) = \frac{n(n+4)(n+5)}{3}$$

13. $$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 7 + \cdots + n(n+1)(2n+1) = \frac{n(n+1)^2(n+2)}{2}$$

14. $1 \cdot 3 \cdot 4 + 2 \cdot 5 \cdot 6 + 3 \cdot 7 \cdot 8 + \cdots + n(2n+1)(2n+2) = n(n+1)^2(n+2)$

15. $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

16. $$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

17. $$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=n+1$$

18. $$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}\quad \text{for}\quad n\geq 2$$